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RENDICONTO
DELL'ACCADEMIA DELLE SCIENZE
FISICHE E MATEMATICHE

SERIE IV - VOL. LXVI - ANNO CXXXVIII

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Valutazione della resistenza all'azione dell'acqua di materiali da costruzione

Nota di Giuseppe Frigione e Luciano Bonavita¹

Presentata dal Socio Ordinario Riccardo Sersale
(Adunanza del 6 febbraio 1999)

Key words: building matter, durability, gypsum, hydraulicity, leaching, water resistance.

Abstract - A test method able to estimate in short time (24 hours) the ability of a binder to produce manufacts resistant to the solvent action of the water is proposed. The method can be applied also to specimens taken from manufacts opportunity treated.

Riassunto - Si propone un metodo di prova che permette di valutare in tempi rapidi (24 ore) l'idoneità di un legante alla fabbricazione di manufatti resistenti all'azione solvente dell'acqua. Il metodo è applicabile anche a provini prelevati da manufatti opportunamente trattati.

1 - INTRODUZIONE

Sotto la denominazione "materiali leganti inorganici" sono raccolti quei prodotti che, mescolati con acqua, danno origine a paste che induriscono progressivamente ed acquistano con il tempo consistenza lapidea. Comunemente i materiali leganti sono classificati in: leganti aerei e leganti idraulici.

La qualifica di legante aereo indica che essi possono indurire soltanto all'aria e che inoltre il manufatto indurito non resiste all'azione dissolvente dell'acqua.

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D'altra parte, il termine idraulico in origine é stato usato per qualificare i leganti destinati alla costruzione di opere idrauliche, cioè idonei alla fabbricazione di manufatti adatti a resistere il più a lungo possibile in contatto con l'acqua senza deteriorarsi. In seguito, con il progredire delle conoscenze sulla natura e sul comportamento di questi materiali, la definizione di legante idraulico é venuta ad assumere un significato più ampio, che trascende il senso letterale del termine idraulico. Attualmente si definisce tale un legante che indurisce per reazione dei suoi costituenti con l'acqua, dando luogo alla formazione di composti idrati molto poco solubili che, dopo conveniente addensamento, resistono indefinitamente all'azione solvente dell'acqua, sia stagnante che in movimento. In forma più concisa si può affermare che un legante idraulico ha la caratteristica peculiare di generare con l'acqua impasti che induriscono anche se immersi in essa, nonché un prodotto indurito non dilavabile (1).

Tuttavia, attualmente sono descritti in letteratura prodotti che, pur a base di leganti aerei, mediante particolari aggiunte (2-10), risultano resistenti all'azione dell'acqua, pur non essendo in grado di indurire in ambiente subacqueo. Sorge pertanto la necessità di metodi di prova che verifichino il comportamento di tali materiali verso l'acqua.

Invero, Sing *et al.* (11) verificano la resistenza all'acqua del manufatto misurando l'incremento di peso dovuto ad assorbimento a seguito di immersione. Analogamente la Norma ASTM C 630 (12), per misurare la durabilità del cartongesso, prevede misure di assorbimento di acqua. In tal modo si valuta però l'impermeabilità del materiale all'acqua e la sua porosità aperta.

Altri Autori, ad esempio Dass *et al.* (13), al fine di verificare l'idoneità di materiali a base leganti aerei, opportunamente trattati, alla fabbricazione di manufatti che devono resistere il più a lungo possibile in contatto con l'acqua senza deteriorarsi, pongono il manufatto in bagno d'acqua o lo espongono all'atmosfera, verificando che nel tempo (mesi o anni) esso non si deteriori, e controllandone eventualmente il peso, in condizioni standardizzate, prima e dopo il trattamento.

Proponiamo qui di seguito un metodo di prova che permette di valutare in tempi rapidi (24 ore) la resistenza all'azione solvente dell'acqua di provini confezionati con il legante in esame.

2 - METODO PROPOSTO

Il metodo consiste nel sottoporre i provini in esame a pioggia di acqua prodotta con uno spruzzatore del diametro di 3,0 cm con 33 e 22 fori rispettivamente di diametro 1,0 mm, posti su due cerchi concentrici. La pressione nell'impianto è mantenuta costante. L'apparecchiatura è mostrata in figura 1. Essa è costituita dallo spruzzatore, da un manometro per controllare la pressione, e da un rubinetto per regolare la pressione dell'acqua.

Il provino del legante, in formato standard, ad esempio (40 x 40 x 160 mm), dopo condizionamento a peso costante, $\pm 0,5\%$, ad una temperatura di $20 \pm 2^\circ\text{C}$ in atmosfera con U.R. del $50 \pm 5\%$, è pesato con una precisione di 1,0 g. Successivamente viene posto verticalmente sotto lo spruzzatore con la faccia superiore ad una distanza di 24 cm da esso. Dopo 24 ore di trattamento con pioggia, il provino viene rimosso, asciugato e di nuovo portato a peso costante con uguali modalità delle precedenti. Si calcola quindi la perdita di peso del campione come percentuale del peso originale.

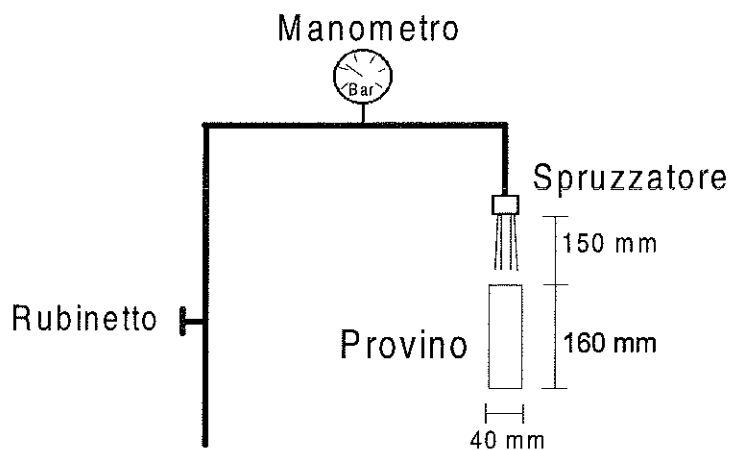


Figura 1: Schema dell'apparecchiatura proposta

Al fine di valutare l'influenza della pressione dell'acqua nell'impianto di prova e di verificare la validità del metodo proposto, una serie di prodotti che notoriamente presentano resistenza all'acqua variabile (2,8,9,10), sono stati sottoposti ad esame sia con un metodo statico, mediante immersione in acqua per 6 mesi, sia con il metodo proposto, operando con due diverse pressioni dell'acqua.

3 - PARTE SPERIMENTALE

I leganti, e le relative miscele, sono state scelti in modo da coprire l'intero intervallo: massima resistenza all'acqua (comportamento idraulico del cemento Portland) - nessuna resistenza all'acqua (miscela 30% calce idrata e 70% sabbia). La lista dei leganti sperimentati é riportata in tabella 1.

Sono state anche preparate miscele costituite da solo Plaster di Paris a rapporti acqua: gesso varianti da 0,30 a 0,70. Ciò al fine di valutare la sensibilità del metodo proposto a determinare la maggiore o minore resistenza all'acqua del materiale, essendo noto che al crescere del rapporto acqua/gesso aumenta la porosità del manufatto. Ne consegue un incremento di superficie di contatto gesso idratato - acqua e una conseguente diminuzione della resistenza all'acqua dilavante.

Con ogni miscela sono stati confezionati nove prismi di dimensioni 40 x 40 x 160 mm che sono stati fatti stagionare per 30 giorni in ambiente con U.R. > 98%, con eccezione delle miscele 20 e 21 (calce idrata + sabbia e argilla) che sono state stagionate all'aria in ambiente laboratorio.

Dopo avere determinato il peso di ogni singolo provino, per ogni miscela tre di essi sono stati posti in acqua stagnante per sei mesi, altri tre sottoposti a prova con il metodo proposto, con pressione dell'acqua di 0,2 e 0,4 bar, rispettivamente.

4 - RISULTATI E DISCUSSIONE

I valori percentuali di perdita di peso - rispetto al peso iniziale - ottenuti con i due metodi (ognuno media di tre determinazioni) sono riportati in Tabella 2 e diagrammati in figura 2 e figura 3, rispettivamente per la pressione di

Tabella 1: Leganti sperimentati.

Miscela No	Composizione
1	Cemento CEN I 32,5 : sabbia = 1:3(14)
2	Calce idraulica acqua/calce 0,60
3	90% calce idrata e 10% pozzolana (fly ash) acqua/miscela = 0,50
4	70% calce idrata e 30% pozzolana (fly ash) acqua/miscela = 0,50
5	50% calce idrata e 50% pozzolana (fly ash) acqua/miscela = 0,50
6	30% calce idrata e 70% pozzolana (fly ash) acqua/miscela = 0,50
7	10% calce idrata e 90% pozzolana (fly ash) acqua/miscela = 0,50
8	Plaster di Paris (di seguito indicato come gesso) acqua/gesso = 0,40
9	90% gesso e 10% loppa* acqua/miscela = 0,60
10	70% gesso e 30% loppa* acqua/miscela = 0,60
11	50% gesso e 50% loppa* acqua/miscela = 0,60
12	30% gesso e 70% loppa* acqua/miscela = 0,60
13	90% argilla** e 10% cemento Portland acqua/miscela = 0,60
14	70% argilla** e 30% cemento Portland acqua/miscela = 0,60
15	95% gesso e 5% argilla (montmorillonite) acqua/miscela = 0,40
16	90% gesso e 10% argilla (montmorillonite) acqua/miscela = 0,38
17	85% gesso e 15% argilla (montmorillonite) acqua/miscela = 0,36
18	80% gesso e 20% argilla (montmorillonite) acqua/miscela = 0,45
19	75% gesso e 25% argilla (montmorillonite) acqua/miscela = 0,50
20	30% calce idrata e 70% sabbia acqua/miscela = 0,40
21	100% argilla** acqua/argilla = 0,45

* basica granulata d'altoforno macinata a 400 m²/kg; ** montmorillonite

Tabella 2: Valori delle perdite di peso ottenute con i diversi metodi.
(Percentuali rispetto al peso iniziale)

Miscela No	% Perdita di peso metodo statico	% Perdita di peso metodo proposto (pressione acqua 0,2 bar)	% Perdita di peso metodo proposto (pressione acqua 0,4 bar)
1	0	0	0
2	1,9	1,8	2
3	43,7	46	52
4	0,2	0,2	0,4
5	0,1	1	4
6	0	2	4
7	0	0,1	0,2
8	70	72,1	74
9	43	58	71,5
10	2	4	14,1
11	3	5	10
12	0	2,5	4
13	5,6	1,5	2
14	0	0	0
15	20	39	49,6
16	25	47,2	58,4
17	26	50,2	58,2
18	36	52	67,5
19	44	49,6	62,2
20	100	100	100
21	100	100	100

esercizio di 0,2 e 0,4 bar: in ascissa i valori relativi al metodo statico ed in ordinata quelli relativi al metodo proposto.

I coefficienti di correlazione sono 0,96947 e 0,93348 - rispettivamente per pressione di esercizio di 0,2 e 0,4 bar -. Ciò indica, essendo uguale nelle due serie di prove il numero di osservazioni, che le prove condotte con pressione dell'acqua a 0,2 bar forniscono valori più rispondenti a quelli del metodo statico. Probabilmente, con la pressione dell'acqua più elevata, influisce anche la resistenza meccanica del materiale.

Alla luce di quanto sopra si può quindi affermare che il metodo di prova proposto - con pressione di lavoro dell'acqua a 0,2 bar - è idoneo a verificare la resistenza all'acqua di un manufatto confezionato con un determinato legante e ciò in tempi molto brevi (24 ore).

In figura 4 è riportato, in funzione del rapporto acqua/gesso, il comportamento dei provini all'azione dilavante dell'acqua, saggiati con il metodo rapido proposto. Si nota l'ottima sensibilità del metodo atto ad evidenziare le variazioni di durabilità anche per piccoli rapporti acqua/gesso. L'andamento iperbolico al crescere del rapporto acqua/gesso è in relazione all'incremento della permeabilità del legante indurito al crescere del contenuto d'acqua (15).

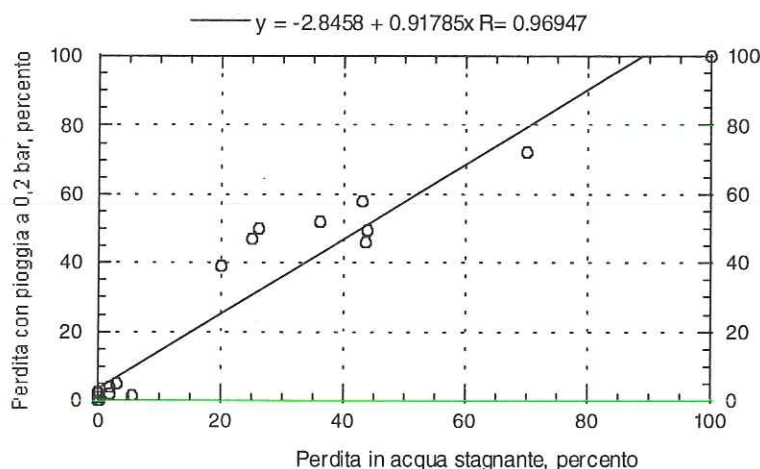


Figura 2: Relazione fra valori di resistenza all'acqua ottenuti con il metodo statico ed il metodo proposto con pressione dell'acqua di 0,2 bar.

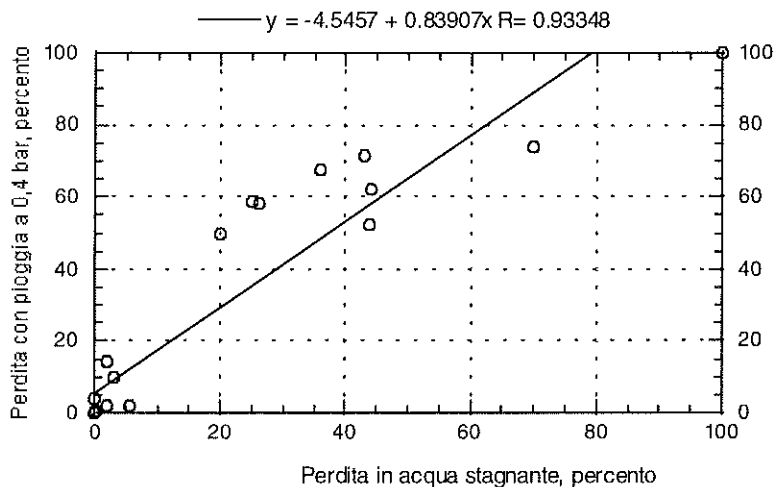


Figura 3: Relazione fra valori di resistenza all'acqua ottenuti con il metodo statico ed il metodo proposto con pressione dell'acqua di 0,4 bar.

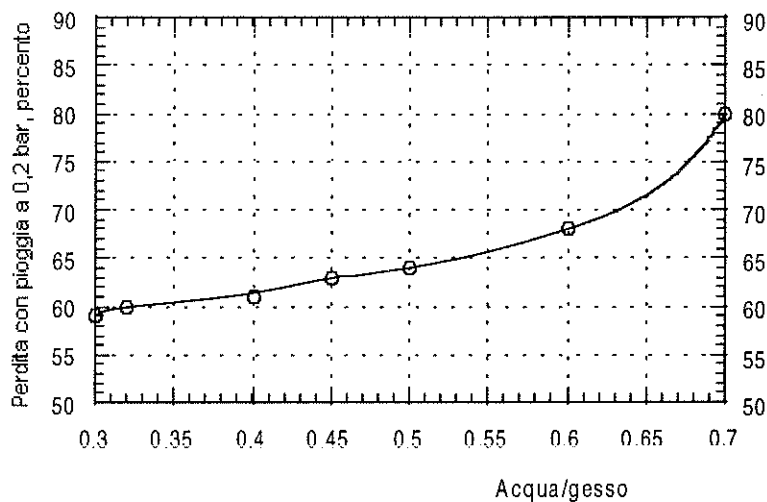


Figura 4: Resistenza all'acqua di manufatti di gesso a vari rapporti acqua/gesso misurata con il metodo proposto (pressione dell'acqua di 0,2 bar).

5 - CONCLUSIONI

L'insieme delle evidenze sperimentali indica che utilizzando il metodo a pioggia da noi descritto è possibile determinare, in tempi ristretti, l'idoneità di un legante alla fabbricazione di manufatti resistenti all'azione dell'acqua. Il metodo è applicabile anche a provini prelevati da manufatti opportunamente trattati.

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Role of square planar transition complexes in homogeneous catalysis of organic reactions

Nota di Ida Orabona¹, Francesco Ruffo^{1*}, Antonio Saporito¹, Diego Tesauro¹

Presentata dal socio Achille Panunzi
(Adunanza del 6 marzo 1999)

Keywords: catalysis, square-planar complexes

Abstract- It is given a survey of the role of transition metal square planar complexes in the catalytic promotion of organic reactions. Those processes are taken in account in which at least one complex with the title geometry is involved in the credited catalytic cycle. The survey, which mainly deals with d^8 ions, summarizes the relevant features which cause the peculiar versatility of the square-planar species and quotes all the more relevant processes, giving an example for each one. Two of the cited examples stem from the activity of the research group of the authors.

Riassunto - Viene illustrato il ruolo dei complessi quadrato-planari dei metalli di transizione nella promozione catalitica di reazioni organiche. Si fa riferimento a quei processi in cui il ciclo catalitico considerato operante prevede l'intervento di almeno una specie con la suddetta geometria. La trattazione, che principalmente verte sugli ioni d^8 , riepiloga i motivi principali della peculiare versatilità dei complessi quadrato-planari, e illustra tutti i principali processi, citando almeno un esempio significativo per ciascuno. Due di tali esempi derivano dall'attività di ricerca del gruppo di cui gli autori fanno parte.

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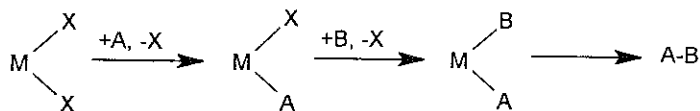
Catalysis, both homogeneous and heterogeneous, is the key to attain most precious chemical productions. Catalysts can range from very simple species, such as proton, to huge molecular frameworks, such as some metallo-enzymes, to solid surfaces. This survey deals with homogeneous catalytic processes prompted by transition metal complexes, which give rise to organic products. The importance of these processes in broadly different areas can be easily enlightened by considering that life-essential enzymes often have metal active centers and that metal-based homogeneous catalysts produce organic materials at a rate of the order of 10^7 tons per year (1990) in the U.S. only. The more relevant features that need to be defined in order to assess the catalyst's efficiency and to properly describe reaction conditions concern the following inter-related aspects:

i) Catalyst to substrate molar ratio; ii) Reaction time; iii) Reaction temperature; iv) Yield.

For a more precise description the results can be given in terms of turnovers, i.e. the number of substrate molecules per catalyst molecule that are converted to product before catalysis stops, or turnover frequency, i.e. the number of moles of product obtained for mole of catalyst per unit time.

In particular the survey is dedicated to those processes whose catalytic cycle is considered to involve at least one square planar complex. A literature search does not disclose any comprehensive review work based on the title point of view, while many more detailed reviews deal with a single ion or a single process. It is to note that the same search enlightens an amazing aspect of the catalytic role of the square-planar complexes. In fact it is found that these species are involved in nearly all the most relevant types of homogeneous metal-promoted processes, with only one relevant fault, that is olefin metathesis.

The reasons for this versatility are obviously related to the general requirements for catalytic ability of coordinated metals. To understand this relationship one can very simply envisage that the metal center typically promotes a cyclic reaction sequence of the type:



where X is a ligand in the precursor species and A and B are added during the cycle. Such a sequence commonly includes also a transformation step (vide infra: *b,c*) of at least one ligand group in the coordination sphere prior to the final product release. Even within this oversimplified view it is apparent that to get an effective catalytic activity both ligand exchange and addition/elimination reactions must be easy processes.

The typical steps which are believed to occur in a catalytic cycle works are essentially coincident with stoichiometric processes attainable on isolable complexes of the same metals. The most common of these reactions are summarized below:

a) Substitution of ligands (including coordinated solvents); *b*) Oxidative addition of a molecule to the metal, with increase of the coordination number and +2 change of the oxidation state; *c*) Nucleophilic addition to a ligand, by outside or by cis migration from another coordination site; *d*) Cis-insertion of one or two atoms in a metal-ligand bond; *e*) Reductive elimination of two ligands to form a coupled species, leaving a metal with coordination and oxidation number diminished by two units.

Reactions of the above types are well suited to the d^8 and other ions affording square-planar species, which are listed in Table 1. In fact it is to note:

- i*) The square-planar geometry allows easy addition of ligand groups, no previous removal of coordinated ligands being in principle required.
- ii*) The stereochemical pattern of the substitution reactions is selectively driven by electronic effects, particularly from trans-effect.
- iii*) The above ions are coordinatively unsaturated in the preferred square-planar environment. Altogether, the attainment of five-coordinate $18e^-$ species for d^8 ions and in general the attainment of higher coordination numbers and of a more populated electron shell does not meet severe energetic hindrance.
- iv*) On the other hand the ions, being fairly electron-rich, can even bear further decrease of the coordination number, increasing their coordinative unsaturation
- v*) Coordination geometries present a good balance between stereochemical selectivity (required for chemo-, regio- and stereoselective catalysis) and stereochemical non rigidity prompting a facile reactivity.
- vi*) The redox chemistry of the ions is amply versatile.

The above observations offer quite sufficient ground for rationalizing the ability of the ions to undergo *a-b* type reactions and analogous steps of catalytic cycles.

A survey of the processes involving square-planar species is given in Table 2. For each type of process the general scheme is given together with one pertaining example. A second example is reported in some cases, dealing with the important class of asymmetric reactions.

Examples, of which two stem from work in our laboratory, are not necessarily recent since they are selected to be representative of the catalyst performance and reaction conditions.

It stems that the square planar complexes, in which palladium compounds show a prominent versatility, are effectively involved in nearly all relevant known processes, although in some cases with poor turnovers. It is certain that future studies will disclose a wider involvement of square-planar species in homogeneous catalysis. In fact, it is to note that several reactions prompted by d^8 ions are not catalytic, nevertheless affording selectively the product(s) in high yield. This is mainly observed in the case the metal is not involved in addition/elimination redox steps and the process simply requires changes in coordination number. A careful choice of the coordination environment can switch the metal activity from stoichiometric to catalytic. Thus, a substantial progress can be predicted, and wider application is expected, e.g. in the promotion of "environment-friendly" chemical productions, such as reactions attainable in water or other harmless solvents.

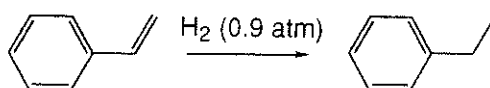
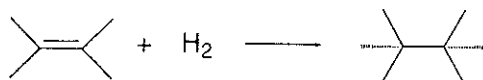
Finally, we note that the above survey could be taken in account by researchers willing to test possible catalytic properties of new classes of square-planar complexes.

Table 1. Electronic configurations of the transition ions
which can afford square-planar complexes

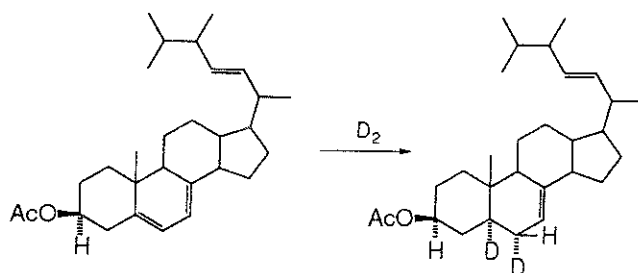
d^8 ions			
Fe(0)	Co(I)	Ni(II)	Cu(III)
Ru(0)	Rh(I)	Pd(II)	Ag(III)
Os(0)	Ir(I)	Pt(II)	Au(III)
d^{10} ions	d^9 ions	d^7 ions	d^5 ions
Cu(I)	Cu(II)	Rh(II)	Mn(II)
	Ag(II)		
	Au(II)		

Table 2. Reactions promoted by square-planar complexes

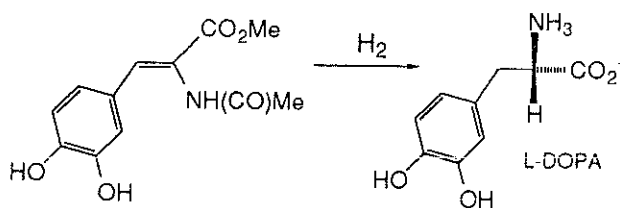
1) Hydrogenation of alkenes



cat.: $[(C_8H_{17})_3NMe]^+[RhCl_4]^-$
J. Org. Chem., 52 (1987) 2804

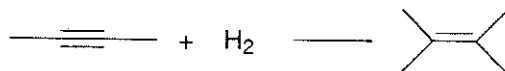


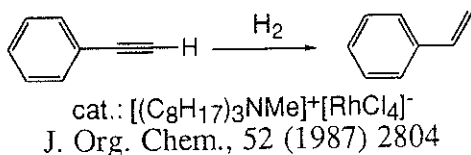
cat.: $[RhCl(PPh_3)_3]$
Org. React., 24 (1976) 1



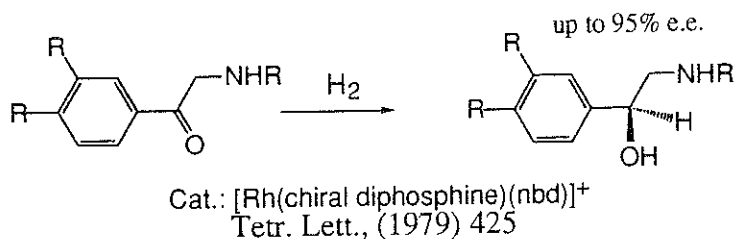
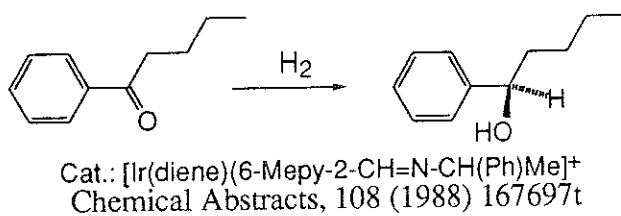
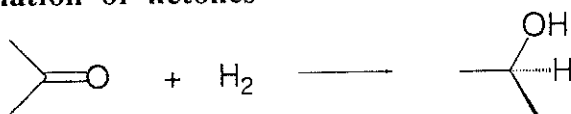
cat.: $[RhL(diop)]^+$
Acc. Chem. Res., 16 (1983) 106 and refs. therein

2) Hydrogenation of alkynes

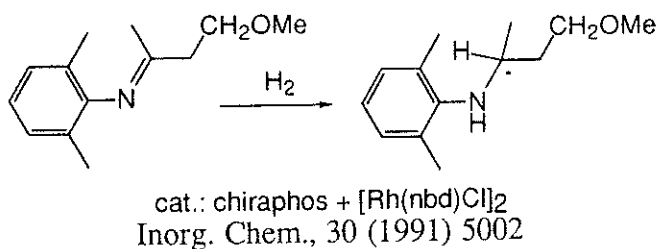
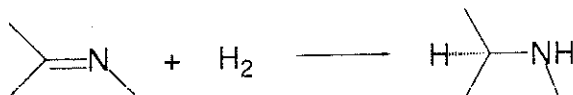




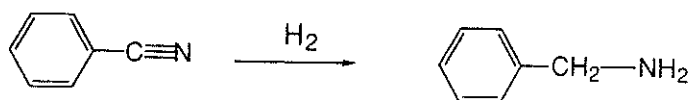
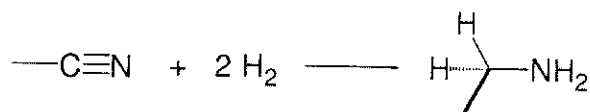
3) Hydrogenation of ketones



4) Hydrogenation of imines



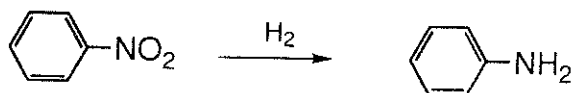
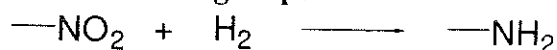
5) Hydrogenation of nitriles



cat.: $[\text{RhH}\{\text{P}(\text{i-Pr}_3)\}_3]$

J. Chem. Soc. Chem. Commun., (1979) 870

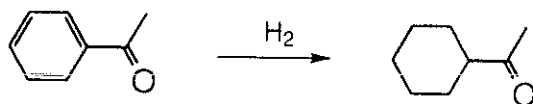
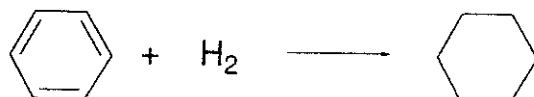
6) Hydrogenation of nitrogroups



cat.: $[\text{RhCl}_3(\text{py}_3)]/\text{NaBH}_4$

J. Chem. Soc. Perkin Trans., (1973) 2509

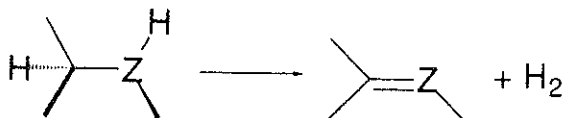
7) Hydrogenation of aromatics

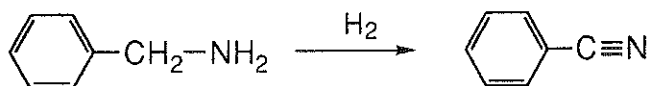


cat.: $[\text{Rh}(\eta^2\text{-C}_5\text{Me}_5\text{Cl}_2)]_2$

J. Chem. Soc. Chem. Commun., (1977) 427

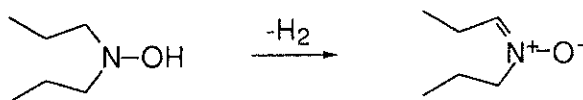
8) Dehydrogenation





cat.: $[\text{RhH}(\text{P}(i\text{-Pr}_3)_3)_3]$

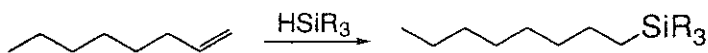
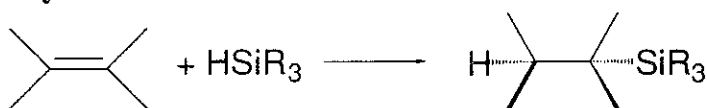
J. Chem. Soc. Chem. Commun., (1979) 870



cat.: $[\text{Pd}(\text{AcO})_2]$

Tetr. Lett., 24 (1983) 1049

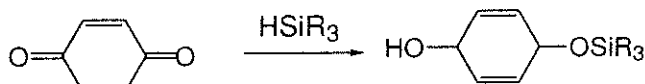
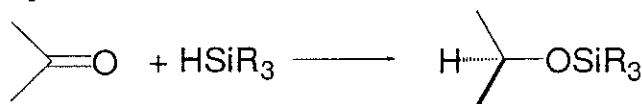
9) Hydrosylation of alkenes



cat.: $[\text{RhCl}(\text{PPh}_3)_3]$

J. Org. Chem., (1984) 3390

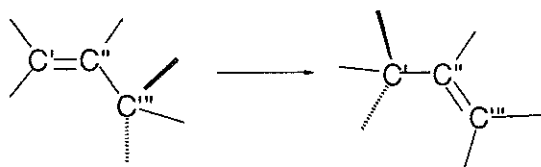
10) Hydrosylation of ketones



cat.: $[\text{RhCl}(\text{PPh}_3)_3]$

J. Organomet. Chem., 308 (1986) C24

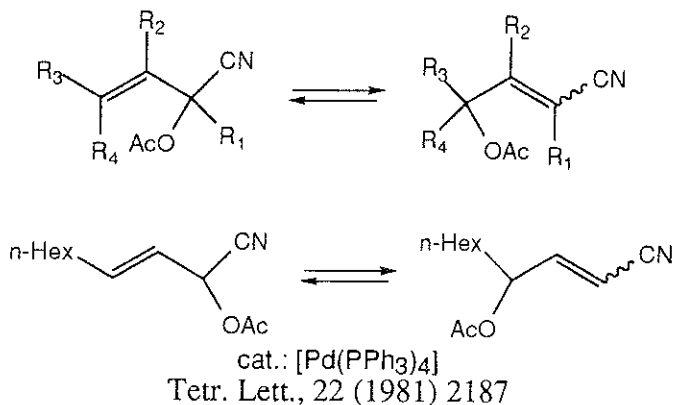
11) Isomerization of alkenes



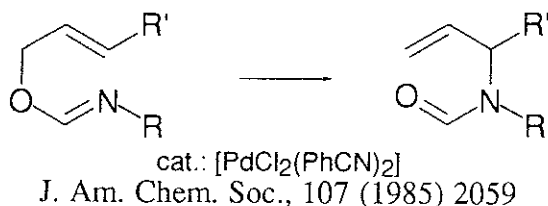


cat.: $[\text{PtCl}_2(\text{PPh}_3)_2]/\text{SnCl}_2$
 J. Am. Chem. Soc., 89 (1967) 3420

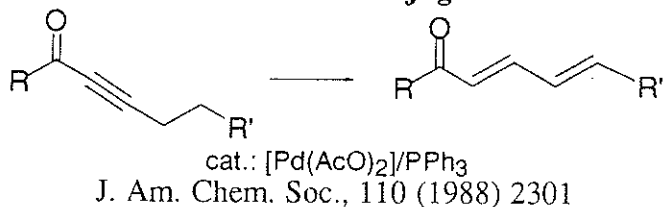
12) Allylic isomerization in the presence of oxygen nucleophiles



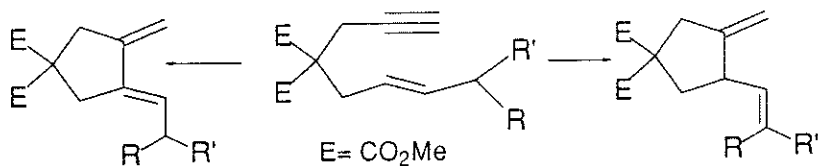
13) Allylic isomerization in the presence of oxygen nucleophiles



14) Isomerization of alkynes to conjugated dienes

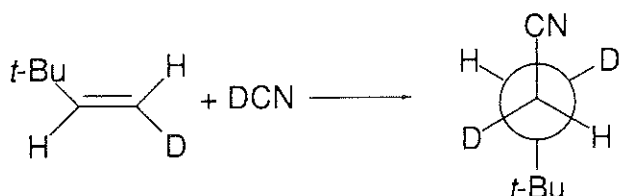


15) Isomerization of 1,6-enyns to cyclic dienes



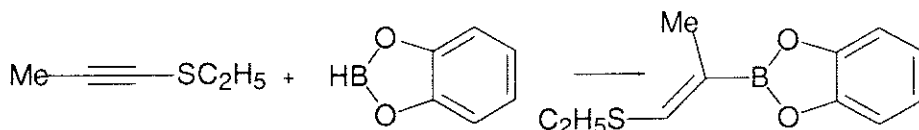
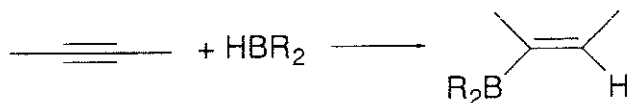
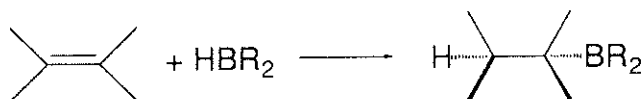
cat.: $[\text{Pd}(\text{AcO}_2)(\text{PPh}_3)_2]$
J. Am. Chem. Soc., 107 (1985) 1781

16) Hydrocyanation of alkenes



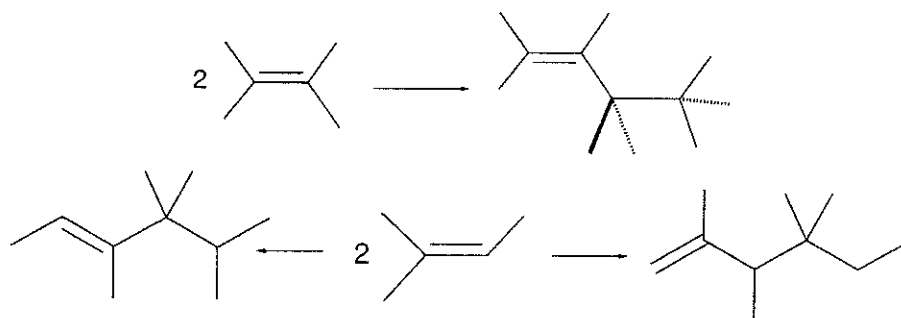
cat.: $[\text{Ni}\{\text{P}(\text{OPh})_3\}_4]$
J. Chem. Soc. Chem. Commun., (1981) 1098

17) Hydroboration of alkenes and alkynes

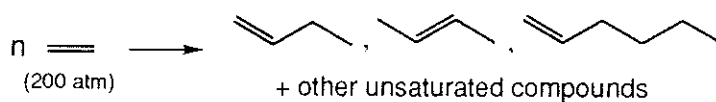


cat.: $[\text{NiCl}_2(\text{diphenylphosphinoethane})]$
Organometallics, 12 (1993) 589

18) Dimerization and related reactions

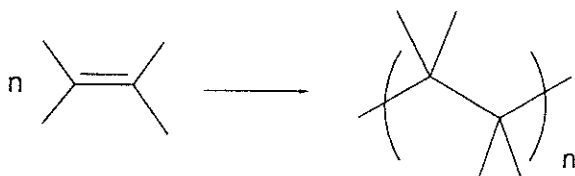


J. Chem. Soc. Chem. Commun., (1976) 47



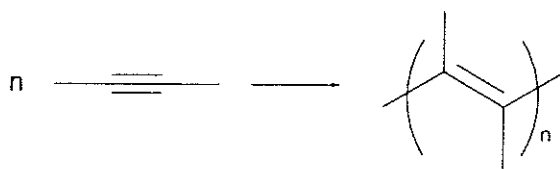
cat.: $[\text{Ni}(\eta^5\text{-C}_5\text{H}_5)(\eta^3\text{-C}_5\text{H}_7)]$
J. Organomet. Chem., 80 (1974) 385

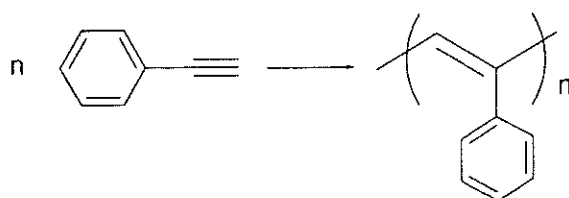
19) Polymerization of alkenes



cat.: $[\text{Pd}(\text{N,N-chelate})(\text{Me})(\text{Et}_2\text{O})]^+$
J. Am. Chem. Soc., 117 (1995) 6414; *ibid.*, 120 (1998) 888

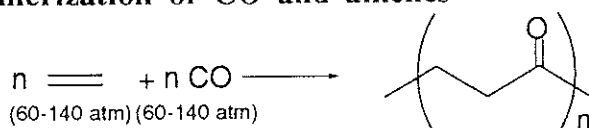
20) Polymerization of alkynes





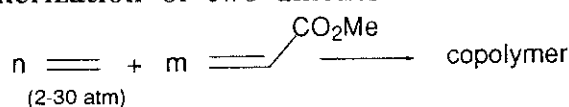
cat.: $[\text{Pd}(\text{MeCN})_4]^{2+}$
Organometallics, 1 (1982) 415

21) Copolymerization of CO and alkenes



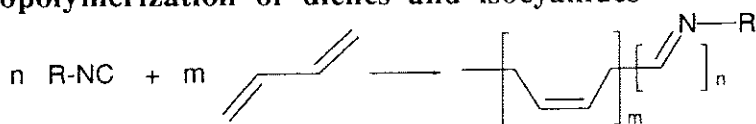
cat.: $[\text{Pd}(\text{MeCN})_4]^{2+}$
Organometallics, 3 (1984) 866

22) Copolymerization of two alkenes



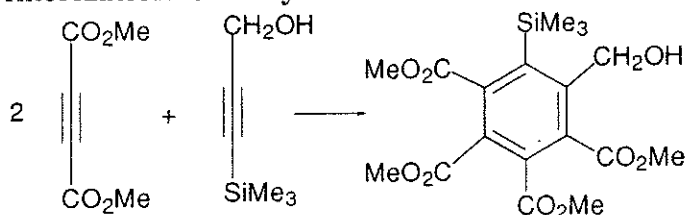
cat.: $[\text{Pd}(\text{N,N-chelate})(\text{Me})(\text{Et}_2\text{O})]^+$
J. Am. Chem. Soc., 120 (1998) 888

23) Copolymerization of dienes and isocyanides



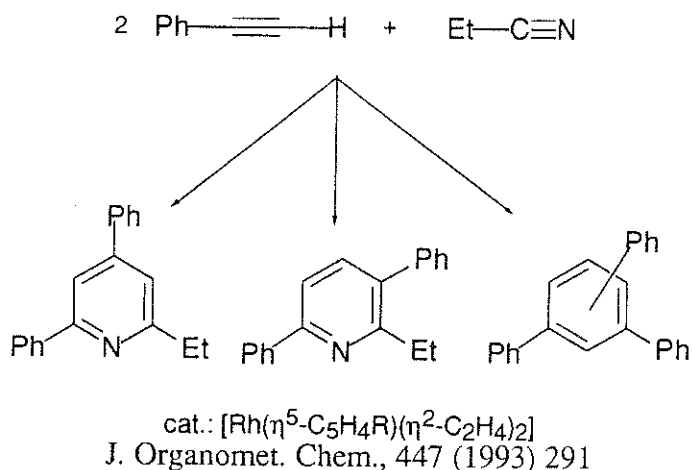
cat.: $[\text{Ni}(\eta^5\text{-C}_3\text{H}_3)(\mu\text{-CF}_3\text{CO}_2)]_2$
Comprehensive Organometallic Chemistry, Pergamon, 12 (1995) 1225

24) Cotrimerization of alkynes

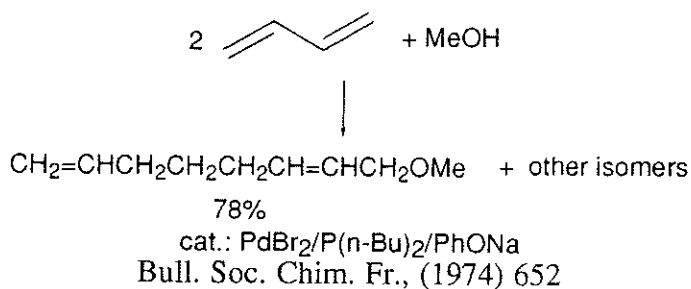


cat.: $[\text{Pd}(\text{N,N-chelate})(\text{C}(\text{CO}_2\text{Me})\text{C}(\text{CO}_2\text{Me})\text{C}(\text{CO}_2\text{Me})\text{C}(\text{CO}_2\text{Me}))]$
J. Organomet. Chem., 384 (1990) 243

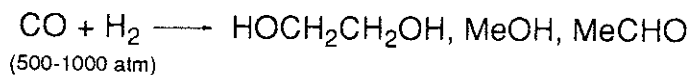
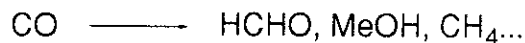
25) Cotrimerization of alkynes and nitriles



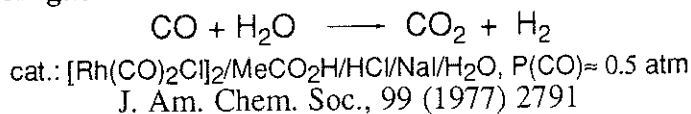
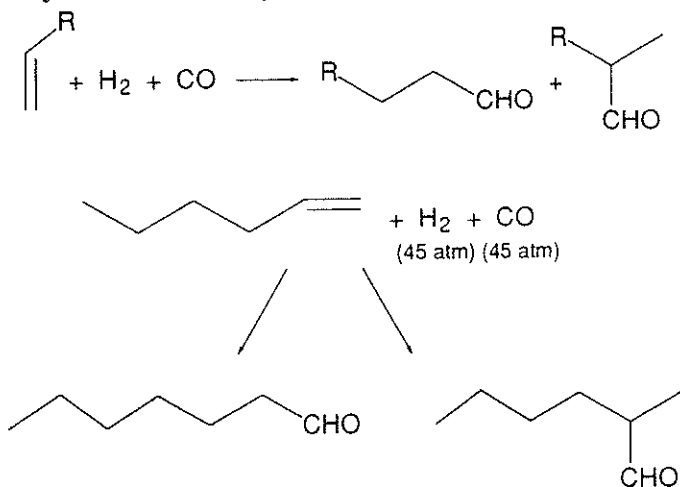
26) Telomerization



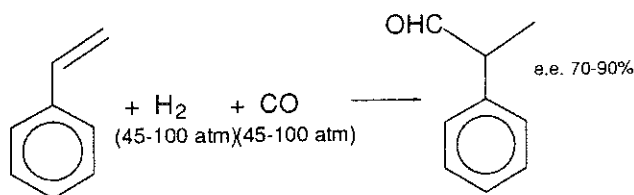
27) CO reduction



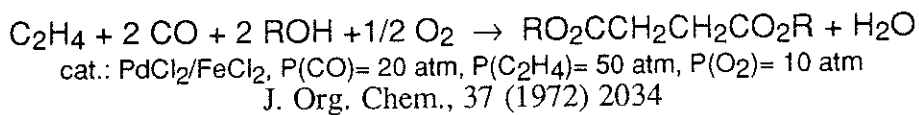
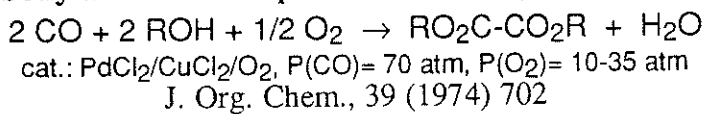
cat.: $[\text{Rh}(\text{CO})_2(\text{acac})]/\text{RCO}_2\text{Cs}$
 Hegedus et al., *Principles and Applications of Organotransition Metal Chemistry*, University Science Book, (1987) 649

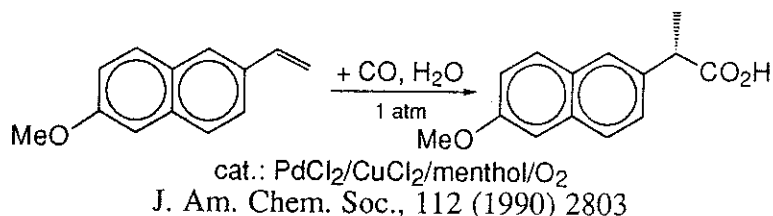
28) Water gas shift**29) Carbonylation in the presence of H_2 : hydroformylation**

cat.: $[\text{Rh}(\text{PPh}_3)_3\text{Cl}_3]$
 J. Chem. Soc. (A), (1970) 1393; Chem. Commun., (1965) 2

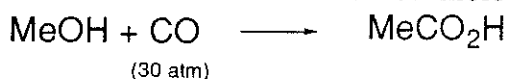


cat.: $[\text{PtCl}_2(\text{DBP-DIOP})]$
 J. Chem. Soc. Chem. Commun., (1982) 473

29) Carbonylation in the presence of H-X , $\text{X} = \text{OH}$, OR , NHR 



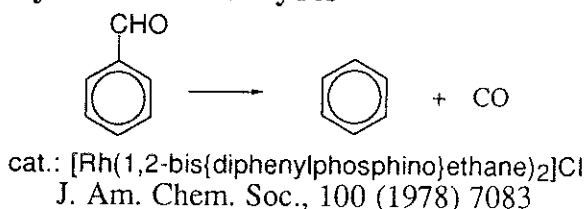
30) Carbonylation in the absence of other molecules



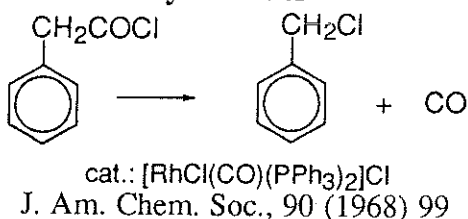
cat.: $[\text{Rh}(\text{CO})_2/\text{I}_2]^+$

Ullmann's Encyclopedia of Industrial Chemistry, Fifth Edition, A1, 45

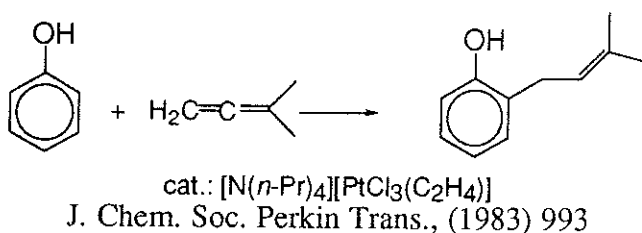
31) Decarbonylation of aldehydes



32) Decarbonylation of acyl halides



33) C-C coupling



M= Sn; cat.: Pd phosphine complexes
Angew. Chem. Int. Ed. Engl., 25 (1986) 508

M= Mg; cat.: [PdCl₂(ferrocenylbis(diphenylphosphine))]
J. Am. Chem. Soc., 106 (1984) 158

M= Hg; cat.: Li₂PdCl₄
J. Am. Chem. Soc., 102 (1980) 2033

M= Tl; cat.: PdCl₂
Bull. Soc. Chim. Jap., 53 (1980) 553

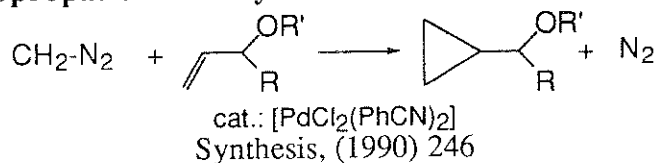
M= Si; cat.: LiPdCl₃
J. Organomet. Chem., 72 (1974) 163

M= Co; cat.: Li₂PdCl₄
J. Chem. Soc. Chem. Commun., (1971) 849

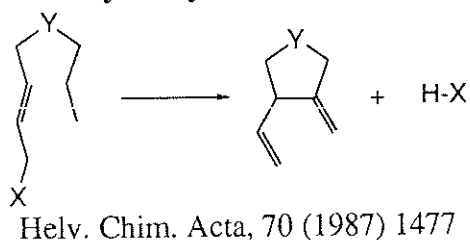
M= Zn; cat.: [NiBr₂P(n-Bu)₃]₂
J. Am. Chem. Soc., 111 (1989) 367

Ar-Y + CH₂=CHR -> ArCH=CHR + [HY] (Heck reaction)
Y= HgR', SnR'₃, PbR'₃, Cl, Br, I; cat.: Li₂PdCl₄/CuCl₂
J. Am. Chem. Soc., 90 (1968) 5518

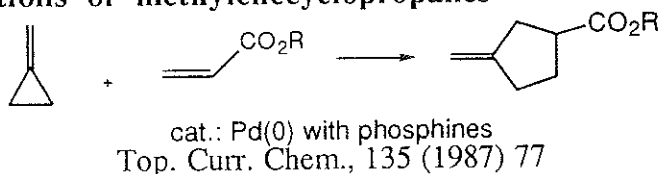
34) Cyclopropagation of allyl derivatives



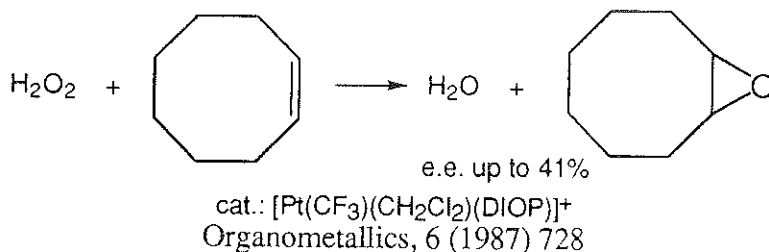
35) Intramolecular allylic cyclization



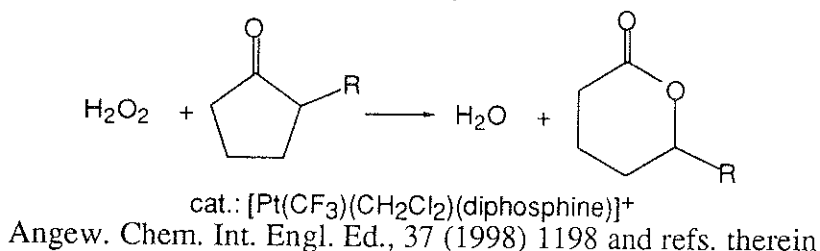
36) Reactions of methylenecyclopropanes



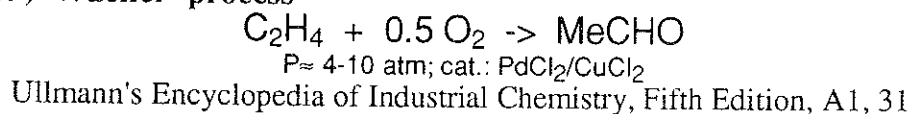
37) Epoxidation of alkenes



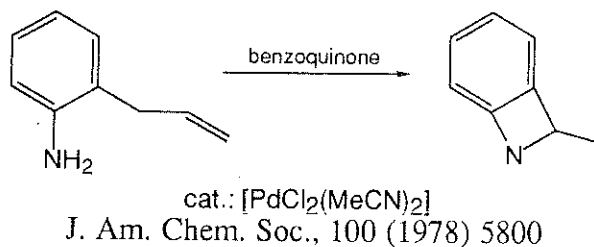
38) Bayer-Villiger oxidation of cyclic ketones



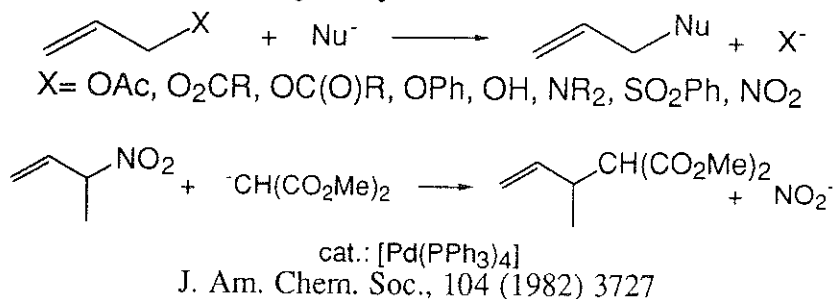
39) Wacker process



40) Amination



41) Substitution in allylic systems



La Teoria delle Congruenze binomie in Italia nei primi trent'anni del secolo XX

Nota di Aldo Scimone*

Presentata dal Socio Luciano Carbone
(Adunanza del 6 marzo 1999)

Key words: number theory, binomial congruences, Galois fields.

Riassunto - Scopo di questo lavoro è quello di esaminare un periodo poco noto, compreso nei primi tre decenni del nostro secolo, della ricerca matematica italiana in teoria dei numeri, relativa alla teoria delle congruenze binomie, cioè alla risoluzione delle equazioni algebriche in un campo di Galois. A questo riguardo, fondamentale fu l'opera di Michele Cipolla (1880-1947). I suoi risultati suscitarono l'interesse di altri matematici italiani per tale argomento, e così contribuirono molto sia a riscattare la teoria dei numeri dal suo ruolo di cenerentola della matematica italiana sia ad incrementare lo sviluppo degli studi algebrici in Italia.

Abstract - The aim of this paper is to examine a little known period, in the first three decades of this century, of the Italian mathematical research on number theory concerning with the theory of binomial congruences, that is, more specifically, with the resolution of algebraic equations in a Galois field. On this subject, the works of Michele Cipolla (1880-1947) were of a fundamental importance. His results aroused the interest of other Italian mathematicians for this subject, and so they largely contributed whether to rescue number theory from its role as Cinderella in Italian mathematics or to increase the development of the algebraic studies in Italy.

1. INTRODUZIONE

In Italia, all'indomani della nascita dello stato unitario vi fu una netta ripresa degli studi matematici. Questo risveglio era iniziato già nel periodo risorgimentale, contemporaneamente al nuovo sviluppo di altre branche della ricerca scientifica. Così, nel 1858 F. Brioschi (1824-1897), E. Betti (1823-

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1892) e F. Casorati (1835-1890), che erano tra i matematici più significativi di quel periodo, intrapresero un viaggio di studio per le università di Gottinga, Berlino e Parigi, segnando l'ingresso della matematica italiana nel contesto europeo. Fu soprattutto per opera loro che la cultura matematica italiana uscì dalle tradizioni erudite di formazione locale di tipo settecentesco o al più ispirate alla cultura francese dell'epoca napoleonica, mettendosi al passo della ricerca europea [Cardamone, 1963], [Bottazzini, 1994].

Ben presto, anche per opera di E. Beltrami (1835-1900), L. Cremona (1830-1903), U. Dini (1845-1918), L. Bianchi (1853-1928) e V. Volterra (1860-1940), sorsero una scuola di Geometria Algebrica e una di Analisi; quest'ultima, in particolare, ebbe le sue premesse nell'opera didattica e scientifica di Betti [Bottazzini, 1982].

Queste due discipline costituirono i principali filoni di ricerca per la matematica italiana fino alla seconda guerra mondiale e anche dopo. Ma, contemporaneamente, seppure in sordina e con pochi cultori significativi, l'interesse degli studiosi si accentuò anche per l'Algebra e la Teoria dei numeri, che in seguito, specialmente nel periodo compreso tra le due guerre mondiali, avrebbero avuto uno sviluppo maggiore. In particolare, per quanto riguarda la Teoria dei numeri, i matematici che vi si dedicarono maggiormente, come A. Genocchi¹ (1817-1889), Bianchi, G. Torelli (1849-1931) ed E. Cesàro (1859-1906) produssero lavori interessanti e in alcuni casi pregevoli, senza che peraltro si riuscisse a creare, per varie ragioni, una vera e propria scuola. Forse, tra le possibili cause di ciò, si potrebbe annoverare innanzitutto la mancanza di una netta autonomia della Teoria dei numeri come disciplina, tant'è che se ne insegnavano alcune parti nel contesto dell'Analisi algebrica. A ciò potrebbe aggiungersi la difficoltà di avere a disposizione, fino al 1880, dei testi in italiano. Infatti, la maggior parte dei trattati erano scritti in francese, in tedesco e in inglese. Ciò comportava un ritardo nella divulgazione della disciplina, specialmente quando la reperibilità delle opere straniere non era agevole.

Nondimeno, sul finire del secolo decimonono, questa situazione precaria cominciò a migliorare. Così, nel 1881 apparve la traduzione dell'opera fondamentale *Vorlesungen über Zahlentheorie* di P. G. L. Dirichlet (1805-1859), con i famosi *Supplementi* di R. Dedekind (1831-1916). La traduzione venne fatta da A. Faifofer (1843-1909) dalla terza edizione tedesca, e la stampa dell'opera rappresentò un evento culturale importante per la comunità matematica italiana. Nel 1895 apparve la traduzione della *Theorie der Congruenzen* di P. L. Tchebychev (1821-1894), ad opera di Iginia

¹ Per una valutazione dell'opera scientifica di Genocchi in teoria dei numeri cfr. [Viola, 1991].

Massarini. Il trattato di Dirichlet comprendeva le lezioni che egli aveva tenuto a Gottinga nel 1855, ma ricalcava essenzialmente l'impianto delle difficili *Disquisitiones Arithmeticae* di C. F. Gauss (1777-1855), e ciò poteva anche scoraggiare chi volesse intraprenderne lo studio. Il libro di Tchebychev era invece più accessibile, perché trattava a fondo la teoria elementare dei numeri, costituita in massima parte dallo studio delle congruenze e nelle linee generali dell'impianto si discostava dalle opere sia di A. M. Legendre (1752-1833) che di Gauss. Infatti, il matematico russo così scriveva nella prefazione [Tchebychev, 1895, pp. X-XI]:

Ma non ostante il gran pregio dell'opera di Gauss, dobbiamo riconoscere che la maggior parte delle sue deduzioni mancano di quella semplicità che distingue i metodi di Euler, di Lagrange e di Legendre. Per tal riguardo non si può preferire lo sviluppo di alcune parti della teoria dei numeri di Gauss a quello delle parti corrispondenti dato da Legendre.

E' chiaro da tutto ciò che nè l'opera di Legendre, nè quella di Gauss danno la teoria dei numeri in quella forma completa nella quale poteva essere presentata dopo le loro ricerche e più ancora dopo quelle dei matematici più recenti. Per questo io non ho seguito, nella compilazione di questa teoria delle congruenze, nè Legendre soltanto, nè solamente Gauss; ma mi sono servito dei lavori d'entrambi e di molti altri matematici che si occupano di quella parte della teoria dei numeri.

Nell'intento della Massarini, come rilevato da lei stessa [Tchebychev, 1895, p. I], il libro si sarebbe dovuto usare come testo nelle scuole superiori:

Il desiderio di vedere diffusa nelle scuole italiane una teoria elementare dei numeri indusse il compianto illustre professore Giuseppe Battaglini a consigliarmi la traduzione della "Teoria delle congruenze" del Tchebychev, "opera classica, mirabile per la scelta delle materie di cui tratta, scritta magistralmente".

Il testo non trattava le parti più elevate della teoria dei numeri, quali lo studio delle forme quadratiche o la teoria degli ideali di Dedekind, ma il suo contenuto era proprio ciò che in quel momento occorreva per coloro che desiderassero prendere confidenza con la disciplina.

Un libro di teoria dei numeri scritto da un autore italiano comparve nel 1897, ad opera di U. Scarpis (1861-1921), con il titolo *Primi elementi della teoria dei numeri*. Studioso stimato e di vasta cultura scientifica, Scarpis² scrisse il volumetto senza alcuna pretesa se non quella di offrire un semplice quadro generale della teoria elementare dei numeri, come ebbe a scrivere egli stesso nella prefazione:

² Per alcune notizie biografiche su U. Scarpis cfr. il necrologio (anonimo) apparso sul Periodico di Matematiche, serie IV, vol. II, 1922, pp. 219-220.

In questa operetta ho esposto con metodo elementare e con sobrietà di dettagli, le più semplici proprietà dei numeri interi. [...] Per dare quindi un esempio della loro grande importanza, e far vedere al tempo stesso come un problema apparentemente di solo carattere geometrico, abbia invece un profondo substrato aritmetico, ho trattato brevemente da ultimo delle equazioni binomie.

Pur avendo un carattere introduttivo, il libro, ispirato al trattato di Tchebychev, possedeva dei pregi espositivi e di impostazione. Per esempio, alla fine di ogni capitolo riportava un buon numero di esercizi per la cui risoluzione venivano offerti spunti e suggerimenti, proprio come viene attuato oggi dalla metodologia della didattica della matematica. Si dovette attendere il 1903 perché uscisse in Italia il primo libro significativo di aritmetica superiore ad opera del matematico cremonese P. Gazzaniga (1853-1930).³ Esso raccoglieva le lezioni di teoria dei numeri che l'autore aveva tenuto nell'Università di Padova fin dal 1885. Nonostante la modestia e il tono dimesso della presentazione dell'opera ai lettori, il testo era molto più di quella «semplice compilazione» dichiarata dall'autore. Innanzitutto, l'elenco delle opere consultate da Gazzaniga comprendeva i lavori più importanti di matematici tedeschi quali Dedekind, P. Bachmann (1837-1920), L. Kronecker (1823-1891), D. Hilbert (1862-1943) e H. Weber (1842-1913), che egli aveva letto direttamente, favorito in ciò dalla padronanza della lingua tedesca.

Inoltre, tutti gli argomenti, dall'analisi indeterminata di primo e di secondo grado alla teoria delle congruenze lineari e di grado superiore, fino alla teoria degli ideali di Dedekind, vi erano esposti con semplicità e chiarezza di stile. Inoltre una delle caratteristiche più salienti del libro consisteva nel far seguire continuamente alla teoria l'appropriata esemplificazione numerica, anche per quelle parti che solitamente non si prestavano agevolmente ad essere illustrate da esempi immediati. Del valore del libro ci si dovette accorgere ben presto. Infatti, Bianchi lo citò nella bibliografia del suo trattato *Lezioni sulla Teoria dei numeri Algebrici*, del 1923, tra le opere raccomandate agli studiosi; e venne pure citato nella bibliografia del trattato di E. Landau (1877-1938) del 1909, *Handbuch der Lehre von der Verteilung der Primzahlen*, segno, questo, che il libro era stato conosciuto ed apprezzato anche all'estero.

Durante questo periodo, a cavallo tra l'ottocento e il novecento, gli argomenti di teoria dei numeri trattati dai matematici italiani furono molto vari. Comunque, furono soprattutto due le parti dell'aritmetica superiore sviluppate maggiormente, la teoria analitica dei numeri (che tratta i problemi della

³ Sull'opera e la vita di Gazzaniga cfr. [Chinello e Emaldi, 1990].

distribuzione dei numeri primi, studia l'andamento delle funzioni numeriche e la teoria dei numeri algebrici e trascendenti) e quella degli ideali.

I risultati più importanti nella teoria analitica furono dovuti a Cesàro e a Torelli. Alla loro opera, e specialmente, come si vedrà più avanti, ad alcune ricerche di Cesàro sui numeri primi, si collegò inizialmente l'attività scientifica di Michele Cipolla (1888-1947), che fu allievo di Torelli. Quest'ultimo aveva insegnato Analisi algebrica a Palermo dal 1891 fino al 1907, contribuendo a creare una scuola di matematici siciliani di grande valore, come, oltre a Cipolla, M. De Franchis (1875-1946), G. Bagnera (1865-1927) e P. Calapso (1872-1934). L'opera più nota di Torelli⁴ in teoria dei numeri fu una poderosa monografia, "extensive report" [Dickson, 1919-23, I p. 440] come la definì il matematico americano L. E. Dickson (1874-1954), sulla totalità dei numeri primi fino a un limite assegnato; un argomento con cui si erano cimentati molti grandi matematici, fin dal tempo di Legendre. La memoria venne premiata nel 1900 dall'Accademia di Napoli, e anche all'estero ebbe una discreta eco, pur presentando qualche imperfezione. [Carbone, Nastasi, Palladino, 1996, pp. 167-177]. Tuttavia, come scrisse Cipolla [Cipolla, 1932, p. 66]:

Se in questo particolare contributo del Torelli qualche risultato non è stabilito con sufficiente rigore (difetto, per altro, comune a procedimenti analoghi di altri Autori, giustificato dalle estreme difficoltà di tali ricerche), ciò nulla toglie all'importanza veramente grande dell'opera. Questa prende in esame circa un centinaio di lavori, non pochi dei quali notevolmente difficili, ed è corredata di grafici e di tavole numeriche per illustrare le formule più notevoli e metterle a raffronto. Essa attesta qual forte fibra di ricercatore fu il Torelli, quale la passione sua, la vastità della sua cultura.

I contributi analitici di Cesàro⁵ furono numerosi. I suoi studi di "aritmetica assintotica" ispirarono i lavori di altri matematici italiani come Torelli, e i suoi lavori sulla distribuzione dei numeri primi, sulle funzioni numeriche e sulla teoria dei numeri algebrici e trascendenti si distinsero per l'originalità dei metodi e la profondità dei risultati [Carbone, Nastasi, Palladino, 1996].

La teoria degli ideali ebbe il massimo cultore in L. Bianchi. Studi recenti [Brigaglia, 1987] hanno messo in luce il ruolo che egli ricoprì nel rinnovamento della matematica italiana tra le due guerre mondiali, e in particolare la sua opera pionieristica nel campo dell'algebra e della teoria dei numeri. Nonostante il fatto che le sue ricerche aritmetiche fossero poco conosciute rispetto al *corpus* maggiore degli altri suoi lavori scientifici, è indubbio che esse contribuirono molto alla diffusione di teorie come quella degli ideali o

⁴ Sulla vita e l'opera di Torelli cfr. [Marcolongo, 1931] e [Cipolla, 1932].

⁵ Per notizie sulla vita e l'opera di E. Cesàro cfr. [Perna, 1907] e [Cerruti, 1907].

dei gruppi di sostituzioni lineari, senza le quali gli studi algebrici non avrebbero potuto avere in Italia una solida base di avvio.

Per quanto riguarda la teoria delle congruenze binomie, alcuni matematici come, per esempio, R. Alagna (1853-1924), N. Amici, U. Scarpis e P. Gazzaniga avevano prodotto alcuni lavori; ma essi riguardavano o la risoluzione di alcuni tipi particolari di congruenze o l'estensione di metodi risolutivi noti. L'unico matematico che aveva fornito alcune formule generali di risoluzione di una congruenza binomia quadratica era stato A. Tonelli (1849-1921), che aveva insegnato Calcolo infinitesimale a Palermo dal 1877 al 1879. Purtroppo, come si vedrà in seguito, le sue formule non si erano rivelate efficaci nelle applicazioni.

Furono solo le ricerche di M. Cipolla a dare nuovo impulso a questo argomento. Egli ne inquadrò la risoluzione nell'ambito più vasto della teoria delle equazioni algebriche in un campo finito, cioè in quella struttura che oggi viene detta *campo di Galois*. Come è noto, l'ordine di un campo di Galois è sempre la potenza di un numero primo p , per cui il campo viene indicato con il simbolo $GF(p^n)$. Ogni $GF(p^n)$ contiene il campo $GF(p)$ e questo è isomorfo al campo delle classi resto dell'anello degli interi modulo p ; per cui la teoria della risoluzione delle equazioni in $GF(p)$ si identifica con quella delle congruenze modulo p . Cipolla si pose il problema di determinare per le congruenze binomie delle formule risolutive sotto la forma di polinomi in cui non vi fossero termini da calcolare per tentativi.

Proprio per sottolineare la struttura, egli chiamò tali formule con il termine "apiristiche" coniato da lui stesso (dal greco α privativo e $\tauειράω$ =tento, provo). In particolare, nel caso della congruenza binomia di tipo generale $x^n \equiv a \pmod{p}$, trovarne una soluzione apiristica significava per Cipolla determinare un polinomio in a che ne fosse una soluzione tutte le volte che a era un residuo n -esimo di p . Le ricerche di Cipolla furono importanti essenzialmente per due motivi. Il primo riguarda i risultati ottenuti che, pur essendo relativi a un settore molto specialistico, si inquadrarono nell'ambito di analoghe ricerche europee [Dickson, 1919-23, pp. 204-222], contribuendo a fare apprezzare all'estero anche questo filone di studi della matematica italiana. Essi costituirono effettivamente un *corpus* teorico importante per la rigorizzazione della teoria delle congruenze binomie, e quindi di quella delle equazioni algebriche nei corpi finiti. Infatti, con la ricerca di soluzioni *apiristiche* l'intento di Cipolla fu quello di eliminare, nella risoluzione delle congruenze, qualsiasi fase intermedia in cui i matematici erano costretti a procedere a tentoni, affidandosi al loro intuito e anche alla fortuna. Inoltre, i suoi lavori furono fonte di ispirazione per altri matematici

che dettero vita ad una vera e propria scuola, favorendo quindi lo sviluppo della teoria dei numeri nell'ambito della ricerca matematica italiana.

2. LA TEORIA DELLE CONGRUENZE BINOMIE

La teoria delle congruenze, che venne sviluppata sistematicamente da Gauss nella sua opera *Disquisitiones Arithmeticae* del 1801 e da lui stesso utilizzata per fondare la teoria classica dei numeri, rimane ancora oggi uno degli strumenti fondamentali per risolvere problemi numerici di carattere teorico. Infatti, l'importanza di questa teoria consiste essenzialmente nel fatto che molti problemi di teoria dei numeri possono essere ricondotti alla questione della risolubilità o meno di qualche tipo di congruenza.

Dopo la pubblicazione delle *Disquisitiones* furono principalmente due le questioni fondamentali che si posero all'attenzione dei matematici: la prima relativa al numero delle soluzioni di una data congruenza algebrica; la seconda riguardante il modo di trovare tutte le soluzioni di una congruenza con un metodo diretto, problema, quest'ultimo, che equivaleva a quello di risolvere le equazioni numeriche mediante formule che dipendessero solo dai coefficienti dell'equazione. Ma i matematici, più che concentrare le loro ricerche sui metodi di soluzione diretta delle congruenze, preferirono dedicarsi maggiormente a migliorare i metodi risolutivi indiretti, cercando, al tempo stesso, di determinare sotto quali condizioni fosse possibile risolvere una data congruenza di grado superiore.

Fra le congruenze, quelle binomie, di forma generale $x^n \equiv c \pmod{p}$, hanno particolare importanza perché permettono di risolvere molte congruenze algebriche, dopo che queste ultime siano state ricondotte al loro tipo. Inoltre, questa teoria ha numerose applicazioni in altre branche dell'Arithmetica superiore, come, per esempio, la ricerca sui divisori dei grandi numeri, specialmente su quelli della forma $a^m - 1$ o $a^m + 1$; la ricerca sui grandi numeri primi e sulla teoria delle equazioni diofantee di grado superiore al primo; problemi nella teoria additiva dei numeri, e così via.

Un criterio valido per stabilire se una data congruenza binomia possa essere risolta o meno, viene fornito dalle cosiddette leggi di reciprocità come, per esempio, quella dei residui quadratici o cubici o biquadratici.

Consideriamo, per esempio, il caso di una congruenza binomia di secondo grado, della forma:

$$x^2 \equiv q \pmod{p}$$

con p numero primo dispari e q non divisibile per p . Se la congruenza ha una soluzione, allora q viene detto un residuo quadratico di p ; se invece la

congruenza non ha alcuna soluzione, q è detto un non residuo quadratico di p . Sono due i problemi fondamentali che dominano la teoria dei residui quadratici: *a)* dato un primo p , determinare quali numeri q siano residui quadratici di p e quali non lo siano; *b)* dato q , determinare i primi p per i quali q sia un residuo quadratico o un non residuo quadratico. Il primo problema era stato oggetto di studio da parte di L. Euler (1707-1783) e di Gauss; invece, la soluzione del secondo problema si basa su un teorema detto *legge di reciprocità quadratica*, che venne stabilita una prima volta da Euler, nel periodo 1744-1746, in una forma non molto chiara [Euler, 1783-85, pp. 64-84], e riscoperta nel 1785 da Legendre [Legendre, 1830, p. 230] che ne diede però una dimostrazione incompleta.

Il primo a darne una dimostrazione completa fu Gauss [Gauss, 1801, artt. 135-141]. Il teorema stabilisce che:

«Se p e q sono primi dispari distinti, allora:

$$(p/q) \cdot (q/p) = (-1)^{(p-1)/2 \cdot (q-1)/2}.$$

Con il simbolo (q/p) , introdotto da Legendre, si denota il *carattere quadratico* di q rispetto a p , per cui il simbolo assume il valore $+1$ o -1 secondo che q è o non è un residuo quadratico di p . Allora il teorema suddetto stabilisce, in altri termini, che il carattere quadratico di p rispetto a q è lo stesso di quello di q rispetto a p quando almeno p o q è della forma $4m+1$; mentre i loro caratteri sono opposti quando sono entrambi della forma $4m+3$. Il teorema viene chiamato *legge di reciprocità* proprio perché stabilisce una reciprocità tra p quale residuo quadratico mod q e q come residuo quadratico mod p .

Per le congruenze binomie di grado superiore il calcolo del relativo simbolo di reciprocità può risultare invece molto difficile se non addirittura impossibile.

In ogni caso, la risoluzione o meno delle congruenze binomie può essere decisa per mezzo del seguente criterio generale, dimostrato pure da Gauss [Gauss, 1801, art. 60]:

«Se

$$x^n \equiv c \pmod{p} \quad (1)$$

è una congruenza binomia di grado n , dove p è un numero primo dispari e c è primo con p , allora essa è risolvibile se e solo se sussiste la congruenza:

$$c^{\frac{p-1}{d}} \equiv 1 \pmod{p} \quad (2)$$

dove $d=(n, p-1)$. Se quest'ultima condizione è verificata, allora la congruenza data ammette d soluzioni che sono tutte quelle della congruenza

$$x^d \equiv c^\lambda \pmod{p},$$

dove λ è un qualsiasi numero che soddisfi alla congruenza

$$(n/d)\lambda \equiv 1 \pmod{(p-1)/d}. »$$

L'importanza di questo teorema consiste nel fatto che esso permette di ricondurre la risoluzione di una congruenza binomia, secondo un modulo primo dispari p , a quella di una congruenza binomia pure di modulo p , ma di grado n uguale a un divisore di $p-1$. Così la condizione (2) di possibilità (dove $d=n$) può essere espressa dicendo che, se verifica la (2), c è chiamato residuo d -esimo di p . Si dimostra pure che in un sistema completo di resti $\pmod{p}$ vi sono precisamente $(p-1)/d$ residui d -esimi di p .

Per il calcolo effettivo delle radici della (1) viene usato un metodo indiretto di soluzione, anch'esso introdotto da Gauss, e basato sulla teoria degli indici.

Infatti, il procedimento più naturale per risolvere la (1) consiste nell'avere a disposizione $(p-1)$ numeri incongrui tra di loro e con lo zero, cioè quello che viene detto *un sistema completo di resti $\pmod{p}$* ; calcolare quindi le potenze n -esime di questi numeri, considerare i minimi resti di queste potenze ed esaminare se uno di essi sia uguale a c . A tal fine, la nozione che gioca un ruolo fondamentale è quella di *radice primitiva di un numero primo p* .

Euler fu il primo a stabilire che per ogni primo p vi fosse qualche numero g il cui ordine era uguale a $p-1$, cioè tale che $g^{p-1} \equiv 1 \pmod{p}$ [con g^h non congruo a $1 \pmod{p}$ per ogni h positivo e minore di $p-1$]. Egli chiamò tale numero una *radice primitiva di p* , e tentò di dimostrare che un numero primo ammette sempre radici primitive, ma fu Gauss [Gauss, 1801, art. 73] a ottenere una dimostrazione di ciò, con un metodo che serve effettivamente a calcolare una radice primitiva di un primo p , e quindi tutte le altre. Infatti, se g è una radice primitiva di p , i numeri $1, g, g^2, \dots, g^{p-2}$ costituiscono un sistema completo di numeri incongrui $\pmod{p}$. Tra questi numeri sono radici primitive di p tutte e sole le potenze di g con esponente che sia primo con $p-1$; quindi, in un sistema completo di numeri incongrui $\pmod{p}$ vi sono $\varphi(p-1)$ *radici primitive di p* , dove $\varphi(n)$ è la funzione aritmetica di Euler, cioè il numero degli interi positivi minori di n e primi con n .

Ebbene, tra i numeri $0, 1, 2, \dots, p-2$ ce n'è uno ed uno solo tale che $g^\alpha \equiv a \pmod{p}$. Il numero α è definito come l'*indice di a rispetto alla radice primitiva g* . Esso viene indicato come $\alpha = \text{ind } a$, per cui si può anche scri-

vere $g^{ind a} \equiv a \pmod{p}$. Fu Gauss a dare la prima trattazione sistematica della teoria degli indici nella terza sezione delle *Disquisitiones*.

Attraverso la definizione di indice, che è chiaramente analoga a quella del logaritmo di un numero rispetto ad una data base, si possono costruire delle tavole di indici anche molto estese. Esse sono utilizzate per risolvere il problema diretto ed inverso: trovare l'indice di un dato numero e viceversa dall'indice risalire al numero.

L'analogia con i logaritmi è ancora più evidente per le proprietà di cui godono gli indici: l'indice d'un prodotto di fattori è congruo alla somma degli indici dei fattori; l'indice della potenza m -esima di un numero è congruo a m volte l'indice del numero, e così via.

La teoria degli indici fornisce non solo la condizione necessaria e sufficiente per la risolubilità di una congruenza binomia secondo un modulo primo, ma serve effettivamente a risolvere la congruenza. Sfortunatamente, per applicare la teoria degli indici è necessario conoscere una radice primitiva del modulo, ma questa ricerca può diventare in pratica ineseguibile quando il modulo è molto grande. È vero che per determinare le radici primitive di un numero primo p si può usare il metodo di Gauss che permette di calcolarle tutte insieme, ma esso è valido solo finché il numero non diventa molto grande. In quest'ultimo caso la ricerca di una radice primitiva di p si effettua elevando a potenza diversi numeri scelti per tentativi. Se fra questi numeri ve ne è uno, δ , tale che nessuna sua potenza minore di $p-1$ risulti congrua ad $1 \pmod{p}$, δ è una radice primitiva. Ad ogni modo, questo metodo può diventare spesso molto laborioso e i calcoli da farsi possono risultare piuttosto prolissi.

Per quanto riguarda le radici primitive di un numero composto m , se a è un qualunque numero primo con m , per il teorema di Euler si ha sempre $a^{\varphi(m)} \equiv 1 \pmod{m}$.

Si dice, allora, che a è una radice primitiva di m , se a^h non è congruo a $1 \pmod{m}$, $\forall h < \varphi(m)$, essendo h un intero positivo. Gauss dimostrò [Gauss, 1801, sez. III] che solo gli interi $2, 4, p^n, 2p^n$ (con n intero positivo e p primo dispari), hanno radici primitive. Anche per determinare le radici primitive di alcuni primi di forma particolare erano stati stabiliti alcuni teoremi, ma in generale rimanevano sempre delle difficoltà. In seguito, la ricerca delle radici primitive fu facilitata dalla pubblicazione di alcune tavole molto ampie come quelle di G. Wertheim⁶, uscite tra il 1893 e il 1899.

⁶ Cfr. *Zeitschrift für Mathematik und Physik*, 25, 1894, pp. 81-97; *Acta Mathematica*, 17, 1893, pp. 315-320; *ibid.*, 20, 1896, pp. 143-152 e 153-157; *ibid.*, 22, 1899, p. 200.

Ciononostante, rimanevano delle difficoltà, se lo stesso Tchebychev scriveva [Tchebychev 1895, p. VI]:

Ed anche oggidì, malgrado tutti i progressi nella teoria dei numeri, o troviamo tutte le radici primitive mediante calcoli troppo laboriosi, o per tentativi, provando diversi numeri, ne troviamo una per dedurne poi tutte le altre [...].

Nella *Appendice II* del suo libro citato, il matematico russo aveva dato alcuni criteri per determinare la radice primitiva di un numero primo quando esso era di forma particolare [Tchebychev, 1895, pp. 209-213]. Una simile lamentela venne espressa anche dal matematico inglese H. J. S. Smith (1826-1883) nel suo magistrale *Report* [Smith, 1894], quando scrisse:

The problem of the direct determination of the primitive roots of a prime number is one of the 'cruces' of the Theory of Numbers.

Per utilizzare la teoria degli indici nella risoluzione della congruenza (1), Gauss, nelle *Disquisitiones*, aveva fornito tavole di indici dei numeri primi minori di 100, e Tchebychev aveva esteso tali tavole fino al numero primo 353. In seguito C. G. J. Jacobi (1804-1851), nel suo *Canon Arithmeticus*, pubblicato a Berlino nel 1839, fornì una tavola di resti e di indici per tutti i numeri primi minori di 1000. Nel 1922 anche A. Cunningham fornì un elenco di radici primitive di tutti i primi fino a 25409 [Cunningham, 1922].

Quando si possiede un elenco di radici primitive, la soluzione della (1) si può ricondurre a quella di una congruenza di primo grado in virtù del seguente teorema dovuto a F. Arndt⁷:

«Data la congruenza

$$x^n \equiv c \pmod{p} \quad (3)$$

dove p è un primo, se una radice primitiva di p è una base degli indici, si ha:

$$\text{Ind } x^n \equiv \text{Ind } c \pmod{p-1},$$

ovvero:

$$n \text{ Ind } x \equiv \text{Ind } c \pmod{p-1}. \text{ »} \quad (4)$$

La congruenza (4) è impossibile se il massimo comun divisore δ di n e $p-1$ non è un divisore di $\text{ind } c$, e in tal caso la (3) non ha alcuna soluzione. Invece, se il massimo comun divisore δ di n e $p-1$ è un divisore di $\text{ind } c$, cioè

$$\text{ind } c \equiv 0 \pmod{\delta} \quad (5)$$

⁷ Cfr. Journal für die reine und angewandte Mathematik, 31, 1846, pp. 333-342.

allora la (4) ha δ soluzioni, per cui $\text{Ind } x$ ha δ diversi valori, e quindi la congruenza (3) ha δ soluzioni.

Per illustrare il metodo supponiamo che sia data da risolvere la congruenza

$$x^8 \equiv 3 \pmod{13}. \quad (6)$$

Applicando il teorema suddetto, essendo in questo caso $\delta = 4$ e prendendo la radice primitiva 2 come una base degli indici, si ha $\text{Ind } 3 \equiv 4$, per cui la condizione (5) è verificata e la congruenza data ha quattro radici incongrue rispetto al modulo 13. La sua soluzione è quindi quella della congruenza

$$8 \text{ ind } x \equiv \text{ind } 3 \pmod{12},$$

cioè:

$$2 \text{ ind } x \equiv 1 \pmod{3}.$$

Otteniamo così $\text{ind } x \equiv 2 \pmod{3}$, ovvero:

$$\text{ind } x \equiv 2, \text{ ind } x \equiv 5, \text{ ind } x \equiv 8, \text{ ind } x \equiv 11 \pmod{12}.$$

I numeri con questi indici sono 4, 6, 9, 7, per cui le soluzioni della (6) sono:

$$x \equiv 4, x \equiv 6, x \equiv 9, x \equiv 7 \pmod{13}.$$

Per quanto riguarda infine la soluzione di una congruenza binomia generale avente per modulo un numero composto, si dimostra il seguente teorema:

«La congruenza binomia generale

$$ax^n \equiv b \pmod{2^\delta p^\pi q^x \dots s^\sigma} \quad (7)$$

non ha soluzioni se una delle congruenze del sistema:

$$\begin{aligned} ax^n &\equiv b \pmod{2^\delta} \\ ax^n &\equiv b \pmod{p^\pi} \\ ax^n &\equiv b \pmod{q^x} \\ &\dots\dots\dots \\ ax^n &\equiv b \pmod{s^\sigma} \end{aligned} \quad (8)$$

non ne ha. Invece, se ciascuna congruenza tra le (8) ha alcune soluzioni, precisamente, n_1 la prima, n_2 la seconda, ..., $n_{\mu+1}$ l'ultima, allora la data congruenza avrà $n_1 n_2 \dots n_{\mu+1}$ soluzioni. Una di esse si troverà determinando un intero x tale che si abbia:

$$\begin{aligned} x &\equiv x_2 \pmod{p^\pi} \\ &\dots\dots\dots \\ x &\equiv x_{\mu+1} \pmod{s^\sigma} \end{aligned}$$

dove $x_1 x_2 \dots x_{\mu+1}$ è un insieme di soluzioni delle (8). »

Al tempo di Cipolla restava aperto, quindi, il problema principale di ottenere le soluzioni di una congruenza binomia in maniera diretta, cioè prescindendo dalla ricerca di radici primitive, senza le quali non si poteva utilizzare la teoria degli indici, che non era applicabile quando il modulo della congruenza era molto grande.

3. L'OPERA DI MICHELE CIPOLLA SULLE CONGRUENZE BINOMIE

Michele Cipolla⁸, oltre ad aver dato notevoli contributi alla teoria dei numeri, fu uno dei maggiori algebristi italiani del suo tempo. Importanti a tale riguardo rimangono i suoi lavori sulla teoria dei gruppi finiti e quelli sulla critica dei fondamenti e sulla storia e la didattica della matematica. [Spagnolo, 1985]. Fu anche un insigne trattatista, e i suoi volumi di *Analisi algebrica*, di *Introduzione al Calcolo infinitesimale*, di *Teoria dei gruppi*, così come le belle *Conferenze di Matematica elementare*, e infine le *Lezioni di Calcolo infinitesimale*, rappresentarono per molto tempo un punto di riferimento per parecchi studiosi.

Per quanto riguarda i lavori di teoria dei numeri, i suoi contributi più significativi apparvero nella prima decade del novecento e furono quasi tutti dedicati al problema di determinare per le congruenze binomie, come s'è detto, delle formule risolutive da lui dette apiristiche. I lavori dedicati a questo tipo di congruenze furono in tutto dodici.

Alcuni suoi lavori aritmetici furono dedicati alla cosiddetta aritmetica asintotica, mentre altri ancora riguardarono questioni aritmetiche varie.

Sulla teoria dei numeri Cipolla scrisse anche un capitolo per l'*Enciclopedia delle matematiche elementari* [Cipolla, 1929a] e un lungo articolo per l'*Enciclopedia Treccani* [Cipolla, 1929b].

⁸ Per notizie sulla vita e l'opera di Cipolla cfr. [Sansone, 1956]; [Amato, 1948]; [Mignosi, 1949].

Prima di parlare dei suoi lavori sulle congruenze binomie, è bene mettere in rilievo il duplice interesse di Cipolla per la teoria dei numeri. Infatti, egli fu attratto sia dall'aspetto analitico della teoria, principalmente da quello relativo ai problemi sulla distribuzione dei numeri primi, sia da quello algebrico centrato sulla teoria delle congruenze. Ciò fu dovuto, forse, principalmente all'influenza di tre matematici che si rivelarono significativi per la sua formazione scientifica, cioè Bianchi, durante il biennio trascorso alla Normale di Pisa, e poi a Palermo G. Torelli e F. Gerbaldi (1858-1934) che fu soprattutto un algebrista e un geometra.

Torelli suggerì a Cipolla l'argomento del suo primo lavoro [Cipolla, 1902], tratto dalla sua tesi di laurea⁹ ampliata e perfezionata in varie parti. Esso trattava il seguente difficile problema: determinare il valore p_n di un numero primo, conoscendo solo il posto n occupato da p_n nella successione dei numeri primi. Questo problema era l'inverso di quello trattato di solito dalla maggioranza dei matematici e che consisteva nel cercare di determinare la totalità dei numeri primi inferiori a un limite assegnato.

L'anno dopo Cipolla iniziò le ricerche sulle congruenze binomie, ritornando all'aritmetica asintotica nel 1905, con un lavoro che estendeva alcuni risultati dei matematici E. Meissel e F. Rogel [Meissel, 1871-1885], [Rogel, 1896-1897] e dello stesso Torelli [Torelli, 1901] sulla totalità dei numeri primi che non superavano un numero assegnato.

Contemporaneamente ai lavori sulle congruenze, Cipolla iniziò le sue ricerche algebriche, e nel 1908, dopo avere risolto il problema delle soluzioni apiristiche per le congruenze binomie, iniziò una serie importante di lavori sui gruppi abeliani e sui gruppi finiti che lo avrebbero imposto ben presto all'attenzione della comunità matematica italiana come "uno dei maggiori algebristi italiani del suo tempo" [Tricomi, 1962, p. 36].

Le prime ricerche di Cipolla sulle congruenze binomie ebbero lo scopo di migliorare alcune formule risolutive o di generalizzare alcuni metodi di soluzione.

Nel suo primo lavoro [Cipolla, 1903] egli trattò la congruenza binomia di secondo grado del tipo:

$$z^2 \equiv q \pmod{p}. \quad (9)$$

Il primo studioso che aveva fornito delle formule generali risolutive della (9) era stato A. Tonelli che in una prima nota lincea [Tonelli, 1892], studiando la congruenza della forma:

$$x^2 \equiv c \pmod{p^\lambda}, \quad (10)$$

⁹ Cfr. Fascicolo n. 3283/1902 dell'Archivio dell'Università di Palermo.

con p numero primo dispari della forma $2^s\alpha+1$, e λ qualunque, aveva dimostrato che le sue radici potevano essere espresse dalla formula generale:

$$x \equiv \pm g^{\gamma\{\varepsilon_0+2\varepsilon_1+\dots+2^{s-2}\varepsilon_{s-2}\}} c^{\frac{\gamma+1}{2}} \pmod{p^\lambda},$$

dove g era un non residuo di p ed $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{s-2}$ dei numeri che potevano assumere solo uno dei due valori 0 e 1. Ma, come egli stesso osservava, la sua formula aveva un valore più teorico che pratico poiché, anche quando si era riusciti a trovare un non residuo g , per ottenere le radici della (10) bisognava calcolare successivamente i numeri $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{s-2}$, nell'ordine in cui erano scritti, o calcolare per tentativi la somma $\varepsilon_0 + 2\varepsilon_1 + \dots + 2^{s-2}\varepsilon_{s-2}$ e tutto ciò non era per nulla agevole.

Il contributo di Cipolla consistette nel trovare una formula risolutiva della (9) che, pur continuando a contenere, come le formule di Tonelli, un elemento che bisognava determinare caso per caso, provando diversi numeri, in generale riduceva di molto i tentativi da farsi. Infatti, la formula che Cipolla proponeva era la seguente:

$$\pm v_{\frac{p+1}{2}} \equiv \pm \frac{\left(k + \sqrt{k^2 - q}\right)^{\frac{p+1}{2}} + \left(k - \sqrt{k^2 - q}\right)^{\frac{p+1}{2}}}{2} \pmod{p}$$

dove $k^2 - q$ era un non residuo quadratico di p , appartenente alla successione:

$$-q, 1^2 - q, 2^2 - q, \dots, \left(\frac{p-1}{2}\right)^2 - q.$$

Il solo numero che bisognava ricercare per tentativi era appunto un non residuo quadratico di p .

Nello stesso anno egli si ispirò ad un lavoro [Alagna, 1899] sulle congruenze dovuto a R. Alagna (1853-1924) che fu tra i fondatori del Circolo matematico di Palermo [Brigaglia e Masotto, 1982], e che oltre ad occuparsi di teoria dei numeri, si interessò anche a quella delle forme e alla teoria dei gruppi. Alagna aveva trattato la risoluzione di una congruenza binomia rispetto ad un numero della forma $2q+1$ e $4q+1$, con q numero primo.

Nella sua ricerca, Cipolla considerò il caso in cui il modulo della congruenza binomia fosse un numero della forma $2^m q+1$ ($m>0$, q primo dispari, non esclusa l'unità). Cipolla elencò innanzitutto alcuni teoremi sulle radici primitive dei numeri di questa forma $2^m q+1$ e quindi fornì delle condizioni necessarie e sufficienti per la loro primalità. Wertheim aveva anche pubblicato a varie riprese delle tavole delle più piccole radici primitive dei numeri primi della forma suddetta e minori di 10000. In base ad alcune

considerazioni sulle radici primitive di questi numeri, Cipolla esaminò le soluzioni delle congruenze del tipo $x^{2'} \equiv N \pmod{p}$ e $x^{2'q} \equiv N \pmod{p}$, dimostrando che a tali congruenze poteva sempre ridursi la congruenza binomia di grado n della forma $x^n \equiv N \pmod{2^m q + 1}$ nel caso in cui essa fosse possibile. Infine, basandosi su questi risultati, generalizzò un risultato di Alagna relativo alla congruenza $q^q \equiv 1 \pmod{2^m q + 1}$.

Dopo questi lavori la sua attenzione venne attratta da una questione teorica importante [Cipolla, 1904a] che riguardava l'esistenza di infiniti numeri composti P che potessero verificare la congruenza di Fermat $a^{P-1} \equiv 1 \pmod{P}$, e che oggi sono chiamati numeri *pseudoprimi*.

Ricordiamo che un numero composto P è detto *pseudoprimo* quando esso divide il numero $2^P - 2$.

In passato si era creduto che il teorema di Fermat fosse una condizione sufficiente per potere affermare che il numero P fosse primo, e quindi che esso potesse servire in generale alla ricerca dei numeri primi; mentre solo in casi assai speciali, quando venivano assegnate condizioni molto particolari alla forma del numero P , l'applicazione del teorema di Fermat, rispetto ad una base scelta convenientemente, poteva fornire una risposta certa sulla natura del numero stesso. Prima di Cipolla, altri matematici, come per esempio Lucas, avevano fornito dei controesempi in proposito, ma fu Cipolla a dimostrare per primo il risultato ricordato. Egli lo dedusse dal seguente teorema:

«Se p è un numero primo dispari che non divide né $(a-1)$ né $(a+1)$, il numero $\frac{a^{2p}-1}{a^2-1}$ e un suo divisore qualunque dispari, verificano la congruenza di Fermat a base a .»

Cipolla basò la dimostrazione di questo teorema su alcune proposizioni relative al *gaussiano* che egli aveva dimostrato nella prima parte del lavoro.

In seguito, la prima dimostrazione induttiva dell'esistenza di infiniti numeri pseudoprimi fu data nel 1948 dal grande matematico polacco Wacław Sierpinski (1882-1969), uno dei maggiori esperti di teoria dei numeri del nostro secolo [Rotkiewics, 1972].

Oltre a questo, Cipolla ottenne altri risultati interessanti. Così, per esempio, escogitò un metodo da applicare in generale alla ricerca dei numeri composti P che verificassero la congruenza di Fermat, qualunque fosse la base della potenza; e fornì pure i criteri, quando P era un numero pari, per riconoscere se la congruenza di Fermat fosse possibile con basi diverse da 1, e nel caso che lo fosse, per trovarne le radici.

A questo lavoro si riferì nel 1912 il matematico americano R. D. Carmichael (1879-1967) il quale estese i risultati di Cipolla per dimostrare che vi erano valori del numero composto P per i quali la congruenza di Fermat risultava vera quando la base era un qualsiasi numero primo con P [Carmichael, 1912].

Nel 1904 Cipolla tornò sul problema della risoluzione delle congruenze binomie di secondo grado rispetto al modulo p^λ della forma $z^2 \equiv p \pmod{p^\lambda}$, con una nota [Cipolla, 1904b] nella quale, sfruttando le proprietà delle funzioni numeriche del secondo ordine, studiate da Lucas, determinò delle formule di risoluzione le quali, per $\lambda = 1$, si riducevano alle formule da lui già determinate nel primo dei suoi lavori. Tra queste formule, la più notevole era la seguente:

$$z \equiv \pm q^{\frac{1}{2}(p^\lambda - 2p^{\lambda-1} + 1)} \frac{\left(k + \sqrt{k^2 - q}\right)^{\frac{1}{2}p^{\lambda-1}(p+1)} + \left(k - \sqrt{k^2 - q}\right)^{\frac{1}{2}p^{\lambda-1}(p+1)}}{2} \pmod{p^\lambda}$$

dove q era un residuo, e $k^2 - q$ un non residuo quadratico di p .

Le funzioni numeriche del secondo ordine utilizzate erano le funzioni simmetriche

$$U_n = \frac{a^n - b^n}{a - b} \quad \text{e} \quad V_n = a^n - b^n$$

delle radici a e b dell'equazione quadratica a coefficienti interi $x^2 = Px - Q$.

Le nuove formule di Cipolla comprendevano anche la formula determinata da Tonelli [Tonelli, 1893]:

$$x \equiv \pm v_0^{2^{\gamma-2}} v_1^{2^{\gamma-3}} \dots v_{s-3}^{2^\gamma} v_{s-2}^\gamma c^{\frac{\gamma+1}{2}} \pmod{p^\lambda}$$

che serviva pure per determinare le radici della medesima congruenza quadratica rispetto al modulo p . Nell'ultima parte della sua nota, Cipolla dedusse pure una formula di risoluzione, sempre per la congruenza binomia quadratica, quando il modulo era un numero composto e dispari.

La teoria delle funzioni numeriche del secondo ordine venne ripresa da lui in una breve nota del 1905 per stabilire una formula che esprimeva le radici della congruenza binomia di secondo e di quarto grado rispetto ad un modulo primo dispari, e che non conteneva alcun elemento da determinarsi caso per caso, provando diversi numeri [Cipolla, 1905a]. L'idea principale si basava su una conveniente trasformazione della somma:

$$\sum_{r=1}^{\frac{p-1}{2}} \frac{x^p - r^p}{x - r} \quad (11)$$

dove x era un numero assegnato della successione:

$$1, 2, 3, \dots, \frac{p-1}{2}.$$

Mediante la (11) Cipolla dimostrò che se la congruenza $z^2 \equiv q \pmod{p}$ era possibile, con p numero primo dispari e q non divisibile per p , allora le sue soluzioni erano date dalla formula:

$$x \equiv \pm 2 (q s_1 + q^2 s_3 + q^3 s_5 + \dots + q^{p-3/2} s_{p-4} + s_{p-2}) \pmod{p} \quad (12)$$

avendo posto

$$s_r = 1^r + 2^r + \dots + \left(\frac{p-1}{2} \right)^r \pmod{p}.$$

La (12), pur avendo valore più teorico che pratico, come rilevato dallo stesso Cipolla, era nondimeno importante perchè era la prima formula apiristica di risoluzione di una congruenza, e costituiva realmente un passo avanti rispetto a tutte le altre formule di risoluzione di una congruenza quadratica ed anche biquadratica. Infatti, come applicazione della (12), Cipolla ottenne delle formule apiristiche della congruenza $x^2 \equiv q \pmod{p^\lambda}$ (quando q era residuo quadratico di p) e della congruenza $x^4 \equiv q \pmod{p}$ quando p era un numero primo dispari della forma $4n+1$.

Nell'ultimo lavoro dedicato alla congruenze binomie biquadratiche Cipolla perfezionò un procedimento escogitato da Legendre [Legendre, 1830], estendendolo opportunamente alla risoluzione della congruenza, supposta possibile:

$$x^{2^m} \equiv a \pmod{2^k}. \quad (13)$$

Legendre si era occupato della risoluzione di tale congruenza nel caso particolare che fosse $m=1$, e a tale scopo aveva escogitato un metodo di risoluzione basato sullo sviluppo di $\sqrt[4]{1+z}$ in serie di potenze di z . La formula di risoluzione a cui era giunto si prestava bene al calcolo effettivo delle radici della congruenza, ma l'unico neo era dovuto al fatto che, nel dimostrare il suo metodo, egli aveva usato alcune serie divergenti in modo non troppo rigoroso, per cui rimaneva qualche dubbio sull'esattezza delle sue formule.

Della risoluzione della (13) si era poi occupato Nicola Amici in una nota del 1894. Egli aveva dimostrato [Amici, 1894] che, se a soddisfaceva a determinate condizioni, la (13) era sempre risolvibile e che a tale scopo si poteva usare il metodo degli indici. Ma per fare ciò erano necessarie apposite tavole di indici, perché egli aveva considerato non gli indici dei numeri rispetto ad una radice primitiva del modulo, bensì gli indici rispetto ad una radice quasi primitiva, chiamando *radici quasi primitive* i numeri inferiori a 2^k della forma $8h+3$ e $8h+5$. Questo metodo non era quindi agevole per valori elevati di k .

Sotto l'ipotesi che a fosse della forma $a=1+2^s A$, con A numero intero dispari ed $s \geq m+2$, Cipolla dimostrò [Cipolla, 1905b] che la (13) ammetteva la soluzione:

$$x_0 \equiv 1 + 2^2 A c_1 - 2^{2s} A^2 c_2 + \dots + (-1)^{n-1} 2^{ns} A^n c_n \pmod{2^k},$$

avendo posto:

$$n = \left\lceil \frac{k-2}{s-m-1} \right\rceil$$

e dove gli c_i erano i coefficienti dello sviluppo in serie di potenze di:

$$\sqrt[2^m]{1+z} = 1 + c_1 z - c_2 z^2 + \dots + (-1)^{n-1} c_n z^n + \dots$$

Nel 1906 i matematici J. Tamarkine e A. Friedmann avrebbero stabilito la seguente formula:

$$z \equiv \pm 2 \sum_{m=0}^{(p-3)/2} q^{\frac{1}{2}(p-1)-m} s_{2m+1}$$

equivalente a quella di Cipolla, nel caso in cui nella (13) fosse $m=1$ e il modulo un numero primo dispari [Dickson, 1919-23, I pp. 219-220].

Dopo questo primo gruppo di lavori, Cipolla, sul finire del 1905, scrisse un primo resoconto sullo stato della teoria delle congruenze [Cipolla, 1905d].

Il resoconto, di carattere storico e bibliografico, fu scritto utilizzando il simbolismo del *Formulaire* di Peano, nel cosiddetto latino *sine flexione*, la lingua che G. Peano (1858-1932) tentava di imporre in quegli anni come strumento di comunicazione scientifica internazionale [Kennedy, 1983]. Esso ebbe il pregio di fornire una ricca documentazione bibliografica che accompagnava le varie parti. Cipolla vi riportò anche i risultati delle sue ricerche, insieme a quelle dei suoi predecessori, per cui il quadro generale della teoria delle congruenze era aggiornato.

Nel biennio successivo l'interesse di Cipolla si volse alla soluzione delle congruenze binomie generali.

Il lavoro più importante per la determinazione di una soluzione apiristica della congruenza binomia generale fu pubblicato da Cipolla nei *Mathematische Annalen* di Lipsia [Cipolla, 1906]. Fu in questa memoria che egli introdusse esplicitamente il termine "soluzione apiristica". Il problema che si propose di risolvere fu il seguente: data una congruenza binomia generale del tipo:

$$x^n \equiv a \pmod{p}, \quad (14)$$

essendo p un numero primo dispari ed a un numero intero arbitrario non divisibile per p , bisognava determinare un polinomio in a , che fornisse una soluzione della congruenza data per ogni residuo n -esimo del modulo p . Il polinomio che egli voleva determinare era della forma:

$$A_0 + A_1 a + \dots + A_{\frac{p-1}{n}-1} a^{\frac{p-1}{n}-1}. \quad (15)$$

Per determinare i coefficienti A_i Cipolla introdusse la nozione di *sistema completo di n -esimo grado (mod p)*, intendendo con ciò $(p-1)/n$ numeri scelti in un sistema completo di resti (mod p), e tali che le loro potenze n -esime fossero tutte incongrue fra loro (mod p).

L'introduzione di tale nozione gli servì per dimostrare il teorema fondamentale:

«Se i numeri

$$r_1, r_2, \dots, r_{\frac{p-1}{n}}$$

formano un sistema completo di n -esimo grado (mod p), posto

$$A_k \equiv -n \left(r_1^{nk-1} + r_2^{nk-1} + \dots + r_{\frac{p-1}{n}}^{nk-1} \right) \pmod{p},$$

allora il polinomio (15) è una soluzione apiristica della congruenza binomia (14).»

Teoricamente il problema era quindi risolto, anche se in pratica il metodo sarebbe stato molto lungo da applicare e avrebbe comportato calcoli faticosi.

Conscio di ciò, Cipolla stesso, nell'ultima parte del suo lavoro, ricavò delle formule risolutive più vantaggiose per le applicazioni, anche se alcuni elementi dovevano essere determinati ancora per tentativi.

Nel 1907 pubblicò un lavoro [Cipolla, 1907a] che si ispirava ad un risultato del matematico tedesco Adolf Hurwitz¹⁰ (1859-1919), relativo alla determinazione del numero delle soluzioni di una congruenza del tipo:

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_rx^r \equiv 0 \pmod{p} \quad (16)$$

rispetto ad un modulo primo p . Il metodo usato da Hurwitz per risolvere questo problema era basato esclusivamente sul teorema di Fermat, e non richiedeva che la sola conoscenza elementare della teoria delle congruenze. Hurwitz aveva dimostrato che il numero delle soluzioni della congruenza (16), sotto l'ipotesi che a_0 non fosse divisibile per p , era uguale al numero intero positivo minore di p , che fosse congruo $\pmod{p}$ alla somma:

$$\sum \frac{(p-1)!}{\alpha_0! \alpha_1! \dots \alpha_r!} a_0^{\alpha_0} a_1^{\alpha_1} \dots a_r^{\alpha_r} \quad (17)$$

estesa a tutte le soluzioni $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_r$, in numeri interi positivi o nulli, dell'equazione:

$$\alpha_0 + \alpha_1 + \alpha_2 + \dots + \alpha_r = p-1 \quad (18)$$

che soddisfacessero alle condizioni:

$$\alpha_0 < p-1, \quad k + \alpha_1 + 2\alpha_2 + \dots + r\alpha_r \equiv 0 \pmod{p-1}. \quad (19)$$

Seguendo lo stesso metodo usato da Hurwitz per giungere a questi risultati, Cipolla determinò dapprima i resti $\pmod{p}$ delle somme delle potenze simili delle soluzioni della congruenza (16). Indicando con s_k la somma delle potenze k -esime delle soluzioni della (16), egli stabilì che tale somma era congrua alla somma (17), tenendo presenti sempre la (18) e la (19). Ciò gli permise di dedurre che il resto $\pmod{p}$ di una funzione simmetrica qualunque delle soluzioni di una congruenza si poteva esprimere mediante una funzione razionale dei coefficienti della congruenza. In particolare, poteva essere determinata la congruenza che ammetteva tutte le soluzioni di quella data e il cui grado fosse uguale al numero delle soluzioni stesse.

Dopo questi risultati interessanti, Cipolla mostrò come il metodo di Hurwitz si potesse estendere per risolvere un'altra questione importante, cioè quella della ricerca delle condizioni necessarie e sufficienti perché due congruenze del tipo:

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_rx^r \equiv 0 \pmod{p}$$

¹⁰ A. Hurwitz, *Über höhere Kongruenzen*, Archiv der Mathematik und Physik, III Reihe, V, 1902, pp. 17-27.

$$g(x) = b_0 + b_1x + b_2x^2 + \dots + b_r x^r \equiv 0 \pmod{p}$$

ammettessero m soluzioni. Egli stabilì che tale numero era dato dalla congruenza:

$$m \equiv - \sum_{i,j} A_j B_i \pmod{p}$$

dove la somma si doveva estendere a tutti i numeri interi i e j che soddisfacessero alle condizioni:

$$0 < i \leq s(p-1), 0 < j \leq r(p-1), i+j \equiv 0 \pmod{p};$$

e dove gli A_j e i B_i erano rispettivamente i coefficienti di $f(x)^{p-1}$ e di $g(x)^{p-1}$.

Nell'ultima parte del lavoro, questi risultati furono estesi da Cipolla al caso generale di n congruenze dei gradi $r_1, r_2, \dots, r_n$ prive di soluzioni congrue a zero secondo il modulo p .

Poco dopo, il matematico americano L. E. Dickson (1874-1954) fornì una doppia generalizzazione della formula di Hurwitz [Dickson, 1919-23, I, p. 232] per il numero delle radici intere della congruenza generale $f(x) \equiv 0 \pmod{p}$ considerando i coefficienti a_i della (16) come elementi di un campo di Galois di ordine p^m .

Sempre nel 1907, due note lincee conclusero questo periodo delle ricerche di Cipolla sulle congruenze binomie.

Nella prima nota [Cipolla, 1907b] egli affrontò la risoluzione della congruenza binomia il cui modulo fosse una potenza di un numero primo p , e il cui grado n fosse una potenza del medesimo numero primo p , cioè della congruenza del tipo:

$$x^{p^r} \equiv a \pmod{p^m} \quad (20)$$

Cipolla applicò a questa congruenza sostanzialmente lo stesso metodo usato per il caso $p=2$, fondato sullo sviluppo in serie di potenze di z della funzione:

$$p^r \sqrt[p^r]{1-z}. \quad (21)$$

Sfruttando il teorema di Fermat-Euler, egli ricavò la soluzione apiristica:

$$x \equiv -a \sum_{n=0}^k c_n \left(1 - a^{1-p^r}\right)^n \pmod{p^m},$$

dove i coefficienti c_i erano dati appunto dallo sviluppo in serie di potenze di z della (21).

Nella seconda nota lineea, egli trattò il caso generale [Cipolla, 1907c] di una congruenza binomia di grado n , secondo un modulo p^m , essendo p un primo dispari ed n un divisore qualunque di $p^{m-1}(p-1)$.

Dopo questa serie di lavori, Cipolla, come già detto, volse la sua attenzione principalmente alla teoria dei gruppi abeliani e a quelli d'ordine finito. Alla teoria delle congruenze non tornerà che nel 1930, quando, sotto la spinta di nuovi e importanti risultati ottenuti in questo campo da altri matematici, dimostrerà l'esistenza di formule risolutive per le congruenze binomie di qualsiasi grado in un corpo finito.

4. I CONTINUATORI DELL'OPERA DI CIPOLLA

I risultati fondamentali ottenuti da Cipolla vennero ripresi nel 1917 da Vincenzo Amato (1882-1963), da poco libero docente di Analisi algebrica nell'Università di Catania. Laureatosi nel 1901, Amato insegnò per un trentennio negli istituti secondari, e nel 1936 vinse la cattedra di Analisi matematica, insegnando prima a Cagliari, poi a Messina, e infine a Catania. Fu essenzialmente un cultore di teoria dei gruppi e di algebra delle matrici.¹¹

In una sua prima nota Amato si propose di determinare, in un corpo quadratico $k(\sqrt{m})$, con m numero intero relativo non divisibile per il quadrato di alcun numero primo, una soluzione apiristica della congruenza binomia:

$$x^n \equiv a \pmod{P}, \quad (22)$$

essendo P un ideale principale primo di 2° grado del corpo, cioè l'ideale determinato dai multipli di un numero primo p , secondo interi del corpo [Amato, 1917a].

Nel caso in cui il modulo fosse stato un ideale primo di 1° grado (per cui la norma $N(P)$ dell'ideale sarebbe stata uguale al numero primo p), una soluzione apiristica della congruenza (22) si sarebbe potuta ottenere dalla corrispondente soluzione, nel corpo razionale, della congruenza rispetto alla norma dell'ideale, come aveva mostrato Cipolla nel lavoro del 1906, apparso nei *Mathematische Annalen*.

Nel caso di un ideale primo di 2° grado, Amato dimostrò che la risoluzione della congruenza (22) si poteva ricondurre a quella di congruenze binomie, secondo il modulo p , nel corpo razionale, aventi per coefficienti numeri noti (e perciò indipendenti da a), le cui soluzioni dovevano pertanto ritenersi note. Fu questa l'idea principale del suo lavoro. Per raggiungere questo scopo, era fondamentale, come nel caso del campo razionale, deter-

¹¹ Per alcune notizie biografiche su V. Amato, cfr. [Dantoni e Tigano, 1963-64].

minare un sistema completo di n -esimo grado ($\text{mod } P$), e ciò poteva essere fatto determinando le radici n -esime dell'unità, secondo il modulo P , cioè risolvendo la congruenza

$$x^n \equiv 1 \pmod{P}, \quad (23)$$

Della (22) Amato fornì una soluzione apiristica nella forma del polinomio

$$A_0 + A_1 a + A_2 a^2 + \dots + A_{\frac{p^2-1}{n}-1} a^{\frac{p^2-1}{n}},$$

nell'ipotesi che fosse soddisfatta la congruenza

$$a^{\frac{N(P)-1}{n}} \equiv 1 \pmod{P}.$$

Nel caso particolare che la congruenza data fosse quadratica, egli mostrò come si potesse determinare un sistema completo di 2° grado ($\text{mod } P$) in grado di fornirne subito una soluzione apiristica nella forma:

$$x_0 \equiv \sum_{k=0}^{\frac{p^2-3}{2}} A'_k a^k \pmod{P}. \quad (24)$$

Nell'ultima parte della nota, egli trattò la questione di determinare in quali corpi quadratici le soluzioni della congruenza (23) potevano essere le radici n -esime dell'unità, e quindi indipendenti dal modulo P , per cui la risoluzione della (22) si sarebbe potuta effettuare qualunque fosse stato il modulo P . Egli trovò che la congruenza (23) ammetteva un sistema completo di soluzioni, costituito dalle n radici n -esime dell'unità, solo nei casi seguenti: quando era $n=2$, in un corpo quadratico qualsiasi; quando era $n=4$, nel corpo quadratico $k(\sqrt{-1})$, cioè dei numeri complessi di Gauss; quando era $n=3$ o $n=6$, nel corpo quadratico $k(\sqrt{-3})$.

C'era da segnalare il dato interessante che per gli ultimi due corpi, dato che in essi sussisteva l'unicità della scomposizione in fattori primi, non era neanche necessario introdurre la nozione d'ideale.

In un lavoro successivo, Amato si propose di semplificare la formula (24), principalmente per ottenere un'espressione più utile per il calcolo delle radici della congruenza quadratica [Amato, 1917b]. Inoltre, le considerazioni che lo portarono ad effettuare la semplificazione desiderata, gli furono anche utili per estendere ad un corpo quadratico i procedimenti già indicati

nel corpo razionale da Cipolla, e ottenere in tal modo delle soluzioni apiristiche della congruenza binomia di grado n .

Un altro matematico che si interessò alle ricerche di Cipolla fu Gaetano Scorza¹² (1876-1939). Figura di spicco fra i matematici italiani che operarono nel periodo compreso tra le due guerre mondiali, di Scorza recenti studi [Brigaglia, 1984, 1987] hanno messo in rilievo l'opera pionieristica nel campo dell'algebra.

Le sue ricerche più importanti riguardarono le cosiddette «matrici di Riemann» (secondo una sua terminologia), cioè le matrici costituite dai possibili periodi indipendenti di una funzione abeliana, ma si interessò pure alla teoria dei gruppi sulla quale lasciò un volume postumo.

Scorza pubblicò nel 1926 una nota [Scorza, 1926] in cui riprese il risultato principale del lavoro di Cipolla pubblicato nei *Mathematische Annalen* [Cipolla, 1906], per mostrare come, attraverso il linguaggio dell'algebra, si potesse giustificare la forma dei coefficienti che intervenivano nella formula di risoluzione apiristica della congruenza binomia considerata. Più precisamente, poiché una congruenza, rispetto ad un numero primo come modulo, non era altro che un'equazione nel corpo numerico finito costituito da un qualsiasi sistema completo di resti rispetto al modulo, egli dimostrò come il problema della risoluzione apiristica si poteva ricondurre a uno d'interpolazione, facendo uso della classica formula di Lagrange in un corpo numerico. Cipolla nel suo lavoro non aveva specificato perché, se il polinomio

$$A_0 + A_1 a + A_2 a^2 + \dots + A_{\frac{p-1}{n}-1} a^{\frac{p-1}{n}}$$

costituiva una soluzione apiristica della congruenza binomia $x^n \equiv a \pmod{p}$, i coefficienti A_k dovessero essere necessariamente della forma:

$$A_k \equiv -n \left(r_1^{nk-1} + r_2^{nk-1} + \dots + r_{\frac{p-1}{n}}^{nk-1} \right) \pmod{p}. \quad (25)$$

Per dare una giustificazione di ciò, Scorza considerò un corpo numerico finito C , con $N=p^m$ elementi, essendo p un numero primo. Dopo di che, dimostrò che risolvere apiristicamente in C l'equazione $x^n - a = 0$ significava costruire un polinomio in C , di grado $q \leq t-1$, con $t = (N-1)/\delta$, e $\delta = (n, N-1)$, del tipo:

$$f(x) = A_0 + A_1 x + A_2 x^2 + \dots + A_{t-1} x^{t-1}, \quad (26)$$

¹² Su G. Scorza si veda il necrologio di F. Severi in «Annali di Matematica», (4) 20, 1941, pp. 1-20, e quello di L. Berzolari nel B.U.M.I., (2), 1, 1939, pp. 401-408.

tale che per ogni a , potenza n -esima di C , $f(a)$ risultasse una radice dell'equazione $x^n - a = 0$.

Scorza mostrò come la determinazione di un tale polinomio si potesse ricondurre ad un problema di interpolazione, e come la (26) potesse scriversi nella classica forma lagrangiana:

$$f(x) = \sum_j^{1..t} r_j \frac{\phi(x)}{\phi'(a_j)(x - a_j)}$$

avendo posto $\phi(x) = (x - a_1)(x - a_2) \dots (x - a_t)$, e indicando con $\phi'(x)$ la derivata di $\phi(x)$.

Sotto questa condizione, i coefficienti della (26) potevano essere posti nella forma:

$$A_h = \frac{1}{t} \sum_j^{1..t} r_j^{1-nh}. \quad (27)$$

Ponendo, in C , $t = -1/\delta$, e cambiando r_j in $1/r_j$, la (27) poteva infine scriversi nella forma:

$$A_h = -\delta \sum_j^{1..t} r_j^{nh-1}. \quad (28)$$

Ebbene, la (28) equivaleva proprio alla formula (25) di Cipolla, supponendo $N = p$, n divisore di $p-1$ e C il corpo numerico costituito da un sistema completo di resti rispetto al modulo p .

Con questo lavoro di Scorza, il risultato di Cipolla veniva così ad essere inquadrato, in modo rimarchevole, nel campo dell'algebra classica.

Il terzo matematico che, ispirandosi all'opera di Cipolla, raggiunse dei risultati significativi fu Gaspare Mignosi.¹³ Vincitore della cattedra di Analisi algebrica nel 1930, dal 1932 insegnò all'Università di Palermo dove, nel 1950, concluse la sua carriera con la nomina a professore emerito. Fu principalmente un cultore dell'analisi algebrica e della teoria dei numeri.

Complessivamente, i lavori di Mignosi riguardarono la risoluzione delle equazioni algebriche in un corpo finito Γ , e si riallacciarono a quelli di Cipolla sulla risoluzione apiristica delle congruenze binomie (*mod* p).

Il suo primo lavoro [Mignosi, 1926] dedicato a questo argomento nacque dalla seguente osservazione: per Scorza era stato possibile applicare con successo la teoria dei corpi numerici al problema della risoluzione apiristica della congruenza binomia $x^n \equiv a \pmod{p}$ solo nell'ipotesi che il modulo della congruenza fosse primo. Infatti, era noto che, sotto questa ipotesi, un

¹³ Per alcune notizie sull'opera e la vita di G. Mignosi cfr. [Mineo, 1952].

sistema completo di numeri incongrui secondo un modulo primo p costituiva un corpo numerico finito rispetto alla relazione di congruenza (*mod* p). Ma quando il modulo era composto non era più possibile utilizzare la stessa teoria perché, in tal caso, un sistema completo di numeri incongrui secondo un modulo composto non costituiva più un corpo numerico. Per affrontare il problema sarebbe stato quindi opportuno considerare un campo di integrità particolare (che nella terminologia di Mignosi equivaleva ad un anello). Però, in tal caso, tra altre difficoltà, per giungere alle formule di risoluzione apiristica delle congruenze binomie considerate, ci sarebbe stata quella di dovere usare speciali sviluppi in serie di potenze di una variabile, di cui già Cipolla aveva fatto uso [Cipolla, 1905b]. Per aggirare tale difficoltà, Mignosi pensò di rappresentare le soluzioni delle congruenze binomie, con modulo e grado potenza di un numero primo qualunque, sotto la forma di serie convergenti in particolari campi d'integrità. A tale scopo, gli fu necessario sostituire all'ordinario concetto di convergenza, basato sulla nozione di continuità, quello di *convergenza in un dato campo d'integrità* che egli definì nel modo seguente.

Sia Γ un campo d'integrità finito rispetto ad una relazione di uguaglianza ed $s_1, s_2, \dots, s_n, \dots$ una successione di numeri di Γ . Se un numero u di Γ è tale che $s_n = u$, per un numero qualsiasi di valori distinti dell'indice n , allora si dirà che u è un *valore limite* della successione $s_1, s_2, \dots, s_n, \dots$ in Γ . Una successione di numeri di un campo d'integrità finito Γ convergerà al numero u in Γ , se u è il solo valore limite che essa ammette in Γ .

Mediante questo nuovo concetto egli determinò una soluzione apiristica della congruenza binomia

$$x^{2^r} \equiv a \pmod{2^\sigma},$$

nella forma:

$$x_0 \equiv c_0 - c_1(a-1) + c_2(a-1)^2 - \dots + (-1)^n c_n(a-1)^n + \dots \pmod{2^\alpha},$$

in cui

$$c_0 = -1, \quad c_1 = \frac{1}{2^r}, \quad c_n = \frac{(2^r-1)(2 \cdot 2^r-1) \dots ((n-1)2^r-1)}{2^m \cdot m!}. \quad (n > 1)$$

Nella parte finale della nota Mignosi trattò il caso più generale di una congruenza binomia in cui sia il modulo che il grado fossero potenza di un numero primo dispari p . Anche in questo caso egli adottò lo stesso procedimento utilizzato nella prima parte del suo lavoro per dimostrare che la congruenza $x^{p^r} \equiv a \pmod{p^\alpha}$, sotto la condizione $a^{p^{\alpha-r-1}(p-1)} \equiv 1 \pmod{p^\alpha}$, ammetteva come soluzione apiristica la somma della serie (convergente in Γ)

$$x_0 \equiv -a \sum_0^{\infty} c_n a^{-p^n} (a^{p^n} - a)^n \pmod{p^\alpha},$$

in cui

$$c_0 = -1, \quad c_1 = \frac{1}{p^r}, \quad c_n = \frac{(p^r - 1)(2 \cdot p^r - 1) \dots ((n-1)p^r - 1)}{p^m \cdot m!}.$$

In altri due lavori successivi, Mignosi affrontò il problema della risoluzione apiristica di un'equazione algebrica in un corpo finito. Nel primo [Mignosi, 1929], riferendosi ad alcuni lavori di Giovanni Sansone sulla risoluzione della congruenza cubica (*mod p*) [Sansone, 1929a], ottenuta mediante un metodo diretto, si propose di mostrare come la risoluzione di una equazione cubica in un corpo finito qualunque Γ , si potesse sempre ricondurre alla risoluzione di equazioni binomie, anche ampliando eventualmente il corpo Γ , trasportando così alle equazioni cubiche di un corpo finito qualcuno dei metodi classici relativi alle equazioni algebriche complete di terzo grado.

Il risultato notevole fu che se tali equazioni binomie, alle quali veniva ricondotta l'equazione cubica data, non erano possibili in Γ , lo erano però sempre nel corpo ampliato Γ' , derivato da Γ mediante il trinomio $\omega^2 + \omega + 1$, con ω radice n -esima dell'unità.

Nel secondo lavoro¹⁴ più ampio e di carattere più generale [Mignosi, 1930a], dimostrò che una condizione necessaria e sufficiente per la risoluzione di un'equazione algebrica in un corpo finito Γ mediante radicali, poteva essere fornita interpretando opportunamente l'equazione stessa nel corpo complesso K , chiarendo in che modo essa potesse essere risolta per mezzo di radicali.

Questa condizione si rivelò importante perché gli permetteva, per i corpi finiti Γ , di utilizzare i risultati classici della risoluzione algebrica delle equazioni. Il risultato principale di Mignosi fu quello di dimostrare che il corpo Γ ammetteva in K infinite interpretazioni, fornendo quindi una risposta definitiva per la risoluzione delle equazioni algebriche dei primi quattro gradi in un corpo finito o in suo conveniente prolungamento.

Nell'ultima memoria dedicata alla teoria delle equazioni algebriche in un corpo finito, Mignosi propose una soluzione alternativa a quella presentata da Cipolla nell'ultimo suo lavoro sulle congruenze, pubblicato nel 1930. In esso Cipolla forniva il risultato principale e teoricamente definitivo a tutte le

¹⁴ Di tale lavoro Mignosi aveva fatto una comunicazione durante il Congresso dei matematici di Bologna del 1928, come appare dagli Atti, tomo II, pp. 81-86.

ricerche sue e dei suoi continuatori [Cipolla, 1930]. Il problema consisteva nel risolvere apiristicamente un'equazione di grado n in un corpo finito C . L'idea di Cipolla fu non solo quella di ridurre il problema ad una questione d'interpolazione, ma di risolverlo in base ad una formula interpolare *propria di un corpo finito*, che era più vantaggiosa di quella classica di Lagrange, sia perché più semplice, sia per altri motivi connessi al problema stesso. A tale scopo, egli introdusse la nozione di *funzione indicatrice* I di un sistema di m numeri in C , definita nel modo seguente. Dato in un corpo finito C , d'ordine P , un sistema di m numeri $\alpha_1, \beta_1, \dots, \gamma_1$, si definisce *funzione indicatrice* del sistema ogni funzione di m variabili $\alpha, \beta, \dots, \gamma$ che assume in C il valore I se a tutte le variabili si danno rispettivamente i valori $\alpha_1, \beta_1, \dots, \gamma_1$, mentre assume il valore 0 in ogni altro caso. Egli indicò una tale funzione come:

$$I(\alpha, \beta, \dots, \gamma; \alpha_1, \beta_1, \dots, \gamma_1) = \{1 - (\alpha - \alpha_1)^{P-1}\} \{1 - (\beta - \beta_1)^{P-1}\} \dots \{1 - (\gamma - \gamma_1)^{P-1}\}$$

e con essa costruì un'opportuna funzione interpolare della forma:

$$\sum_{i=1}^v r_i I(\alpha, \beta, \dots, \gamma; \alpha_1, \beta_1, \dots, \gamma_1),$$

dove gli r_i erano v valori di C .

Allora, se

$$F(x) = x^n + a_1 x^{n-1} + \dots + a_n = 0 \quad (29)$$

era l'equazione di grado n in C , con a_n indeterminato, posto $q = -a_n$ ed $f(x) = F(x) - a_n$, la (29) si poteva scrivere nella forma:

$$f(x) = q. \quad (30)$$

Se, dando a q un valore q_i , non nullo, quest'ultima equazione ammetteva n radici distinte, allora q_i veniva detto un *numero n -esimo rispetto ad $f(x)$* .

Risolvere allora *apiristicamente* la (30) significava trovare un polinomio nell'indeterminata q che fornisse una soluzione della (30) tutte le volte che q era un numero n -esimo rispetto ad $f(x)$. Cipolla determinò una soluzione apiristica della (30), sempre mediante la funzione indicatrice, con il polinomio

$$A(q) = \sum_{i=1}^v r_i I(q, q_i).$$

Esso aveva la proprietà di annullarsi allora e solo quando q non era un numero n -esimo rispetto ad $f(x)$; quindi, tale polinomio caratterizzava i nu-

meri n -esimi rispetto ad $f(x)$; che risultavano, pertanto, come i numeri del corpo che non erano radici dell'equazione $A(q) = 0$.

La soluzione del caso più generale seguiva immediatamente con le stesse modalità. Teoricamente il problema era quindi risolto, ma, come avvertì lo stesso Cipolla, alla fine della sua nota [Cipolla, 1930, p. 206], il calcolo effettivo degli elementi che costituivano la sua formula aveva l'inconveniente della prolissità, per cui:

[...] non sarebbero quindi superflue le ricerche intese a perfezionare la formula (...), o anche informaté ad altri punti di vista, come quelle particolari del Sansone e del Mignosi.

Il punto di vista di Mignosi, al quale si riferiva Cipolla, presentava la caratteristica di richiedere, in luogo di vari sistemi di n -esimo grado utilizzati da Cipolla, solo un unico sistema di numeri, di cui quelli distinti potevano anche scegliersi ad arbitrio nel corpo. A tale scopo, Mignosi introdusse il concetto di *sistema n -esimo* di Γ , e quello di *sistema completo di grado n* di numeri di Γ . Dimostrò quindi che, fissato un sistema completo di grado n di numeri di Γ , e costruiti i relativi sistemi n -esimi di Γ , mediante la formula interpolare:

$$A = \sum_{i=1}^v r_i \left[1 - (a_{i1} - a_1)^{N-1} \right] \dots \left[1 - (a_{in} - a_n)^{N-1} \right],$$

restava definito un polinomio $A = A(a_1, a_2, \dots, a_n) = 1$ nei coefficienti $a_1, a_2, \dots, a_n$ dell'equazione data $f(x) = x^n + a_1 x^{n-1} + \dots + a_n = 0$, che ne costituiva una soluzione apiristica.

Nel campo di queste ricerche sulla risoluzione delle congruenze binomie, anche Giovanni Sansone¹⁵ (1888-1979) apportò alcuni contributi.

Oltre che alla risoluzione apiristica delle congruenze cubiche e biquadratiche, le sue ricerche di teoria dei numeri si estesero anche all'analisi indeterminata, e specialmente allo studio dei sistemi diofantei.

Alla teoria delle congruenze dedicò dieci lavori che furono pubblicati nell'arco di tre anni, dal 1927 al 1930, la maggior parte dei quali furono brevi note o anticipazioni di lavori più estesi. Alcuni lavori estesi furono pubblicati alcuni anni dopo averne dato delle anticipazioni, per cui fra essi vi è una apparente discordanza di priorità degli argomenti trattati. Fra tutti fecero spicco due ampie *Memorie* degli *Annali di Matematica*, una del *Rendiconto dell'Accademia di Napoli* e un'ultima *Memoria Lincea* che chiuse questo ciclo di studi.

¹⁵ Sull'opera di G. Sansone cfr. [Conti, 1981].

Nella prima memoria degli *Annali di Matematica*, Sansone trattò la risoluzione apiristica delle congruenze cubiche [Sansone, 1929a].

Tale argomento aveva destato l'interesse di parecchi matematici europei fin dal tempo di Cauchy, che se ne era occupato fin dal 1829 nei suoi *Exercices de Mathematiques*. Fra i primi matematici italiani che se ne erano interessati vi era stato G. Cordone il quale, in un lavoro del 1895 sulla risoluzione delle congruenze di quarto grado [Cordone, 1895] mediante delle risolventi cubiche, aveva fornito delle dimostrazioni più semplici di alcuni teoremi formulati dal matematico G. Oltramare sulla risoluzione delle congruenze cubiche della forma $x^3 + 3px + 2q \equiv 0 \pmod{\mu}$.

In seguito, anche U. Scarpis si era interessato alla risoluzione di tali congruenze, precisamente in due lavori, il primo del 1911 e il secondo del 1914.

Nel primo [Scarpis, 1911] egli aveva cercato di applicare ad una congruenza cubica la formula di risoluzione cardanica, e aveva constatato che essa si dimostrava illusoria quando la congruenza ammetteva tre radici, mentre risultava efficace se la congruenza ammetteva una sola radice e che inoltre, in certi casi, succedeva il contrario. Nel secondo lavoro, Scarpis aveva ripreso lo studio generale delle congruenze con lo scopo di accennare alle modificazioni che potesse subire la teoria di Galois, quando la si volesse applicare alla risoluzione di equazioni in un campo finito di integrità [Scarpis, 1914].

Sansone, ispirato dai lavori di Scarpis, si prefisse lo scopo di costruire una formula di risoluzione apiristica della congruenza cubica, senza doverla fare dipendere, come invece aveva fatto Scarpis, da quella di congruenze binomie.

Egli affrontò il problema in maniera diretta, osservando innanzitutto che una congruenza cubica si poteva sempre ridurre, con una sostituzione lineare, o alla forma binomia $x^n \equiv a \pmod{p}$, quindi risolubile con la formula apiristica di Cipolla, oppure alla forma:

$$x^3 + ax + a \equiv 0 \pmod{p}, \quad (31)$$

dove a era supposto intero, p primo e diverso da 2 e da 3. Egli determinò innanzitutto il numero dei valori di a per i quali la (31) avesse tre radici incongrue, o una radice oppure nessuna. Passò quindi a determinare la condizione alla quale doveva soddisfare a perché la (31) avesse tre radici incongrue, o una radice o nessuna. Dopo di che determinò la soluzione apiristica della (31) mediante lo stesso metodo di interpolazione usato da Scorza.

Supponendo che le tre radici della (31) non avessero lo stesso carattere quadratico modulo p , egli ottenne la formula:

$$x \equiv (-1)^{\frac{p+1}{2}} D_{\frac{p-3}{2}}(a) / D_{\frac{p-5}{2}}(a) \pmod{p} \quad (32)$$

in cui con la scrittura $D_k(a)$ veniva indicato il determinante ortosimmetrico di Hankel di ordine k .

Il caso in cui le tre radici della congruenza (31) avessero lo stesso carattere quadratico modulo p fu trattato da Sansone in una seconda memoria [Sansone, 1930]. In essa Sansone dimostrò che la congruenza (31) poteva sempre essere trasformata, mediante la sostituzione lineare:

$$x = \frac{-3y}{2y+3}$$

nella congruenza dello stesso tipo:

$$y^3 - \frac{27a}{27+4a}y - \frac{27a}{27+4a} \equiv 0 \pmod{p}$$

le cui radici, però, non avevano più lo stesso carattere quadratico modulo p , per cui essa poteva essere risolta utilizzando la formula (32).

Nel caso particolare in cui il modulo p della congruenza fosse della forma $p = 6h+7$, e nell'ipotesi che le tre radici della congruenza cubica non avessero lo stesso carattere cubico modulo p , egli pervenne ad una nuova formula risolutiva della (31).

Nell'ultima parte della memoria egli prese in esame la risoluzione delle congruenze cubiche i cui moduli fossero potenze di numeri primi, del tipo p^n , e dimostrò che la loro risoluzione si poteva ricondurre a quella di congruenze di grado minore e a quella di congruenze di modulo p^{n-t} con $t > 0$.

Ed allora ne segue che siccome noi sappiamo risolvere le congruenze di secondo grado e quelle cubiche di modulo p , *sappiamo risolvere tutte le congruenze cubiche modulo p^n* . [Sansone, 1930, p. 23]

Sansone fornì nuove formule risolutive [Sansone, 1929b] per la congruenza cubica (31), trattando il caso in cui le tre radici x_1, x_2, x_3 avessero lo stesso carattere quadratico modulo p , mentre non lo avevano le tre differenze $x_1 - x_2, x_2 - x_3, x_3 - x_1$. Egli dimostrò che con una opportuna trasformazione la (31) poteva ricondursi ad un'altra congruenza per la quale sussistevano le relazioni seguenti relative al loro carattere quadratico:

$$\left(\frac{x_1}{p}\right) = \left(\frac{x_2}{p}\right) = \left(\frac{x_3}{p}\right); \quad \left(\frac{x_1 - x_2}{p}\right) = \left(\frac{x_2 - x_3}{p}\right) = \left(\frac{x_3 - x_1}{p}\right).$$

Sotto quest'ipotesi, distinse due casi in cui il modulo p poteva assumere una delle due forme: $p = 4l+1$ oppure $p = 4l+3$.

Nel primo caso, una radice della (31) era fornita dalla formula:

$$x \equiv (-\alpha + 1) \left/ \left(2 \frac{3\alpha^2 + a}{\alpha^3 + a\alpha - a} + 4 \frac{\varphi'(a)}{\varphi(a)} \right) \right. \pmod{p}$$

in cui il polinomio $\varphi(a)$ era definito univocamente da una congruenza differenziale del secondo ordine di tipo fuchsiano.

Nel secondo caso, una radice della (31) era data dalla formula:

$$x \equiv -\alpha + \frac{1}{4} \cdot \frac{\varphi(a)}{\varphi'(a)} \pmod{p}$$

per ciascuno dei $3l$ valori di α che non soddisfacevano la congruenza

$$\varphi(\alpha) \equiv 0 \pmod{p},$$

e che erano distinti da $-x_1, -x_2$ e $-x_3$.

Una sintesi di tutte le ricerche relative allo studio delle congruenze di grado qualunque rispetto ad un modulo primo e alle applicazioni alle congruenze di terzo e di quarto grado venne fornita da Sansone in una estesa memoria lineea [Sansone, 1928].

Nella prima parte della memoria egli studiò le congruenze di grado qualunque rispetto ad un modulo primo. Riguardo a ciò, nel 1902, il matematico G. Rados aveva stabilito un criterio [Rados, 1902] che faceva dipendere il numero delle radici di una congruenza dal valore della caratteristica aritmetica di un determinante di ordine $p-1$, essendo p il modulo.

A questo criterio Sansone ne sostituì uno nuovo che faceva dipendere il numero delle radici della congruenza dalla caratteristica di un determinante di ordine uguale al grado della congruenza stessa. Egli applicò questo criterio per determinare la formula risolutiva di una congruenza di n -esimo grado che ammettesse una sola radice. Precisamente, considerata la congruenza:

$$f(x) = x^n + a_1x + \dots + a_{n-1}x + a_n \equiv 0 \pmod{p},$$

che ammetteva soltanto la radice α , si aveva per questa l'espressione:

$$\alpha \equiv A_{n,n-1} / A_{n,n} \pmod{p}$$

in cui A_{ik} indica il complemento algebrico dell'elemento a_{ik} nella matrice (a_{ik}) ($i, k = 1, \dots, n$). Dopo avere completato questo studio generale delle

congruenze, egli applicò i risultati ottenuti alle congruenze di terzo e di quarto grado. Per la congruenza cubica della forma generale:

$$x^3 + a_1x^2 + a_2x + a_3 \equiv 0 \pmod{p}$$

determinò innanzitutto il numero delle radici, e poi le condizioni necessarie e sufficienti perché essa avesse tutte e tre le radici con lo stesso o con diverso carattere quadratico modulo p . Per la congruenza generale di quarto grado, mediante la risolvente di Euler, determinò innanzitutto le condizioni perché essa avesse quattro radici incongrue. Dopo di che, utilizzando la risolvente di Ferrari, ne determinò le quattro radici, senza la necessità di ricondurne la risolubilità a quella di una congruenza binomia, come invece aveva fatto Cordone [Cordone, 1895].

Nella parte finale della memoria venne esaminato il caso delle congruenze del tipo:

$$x^4 + ax^2 + bx + c \equiv 0 \pmod{p}$$

che ammettessero radici multiple. In particolare, quando una di tali congruenze ammetteva una radice che non aveva il carattere quadratico delle altre tre, Sansone ne determinò la soluzione mediante una formula in cui comparivano particolari determinanti. Egli fornì infine le formule risolutive per le congruenze di quarto grado che ammettevano un numero di radici minore di quattro, portando anche numerosi esempi.

4. CONCLUSIONE

Retrospectivamente, nei primi trent'anni del ventesimo secolo la ricerca matematica italiana in teoria dei numeri si distinse per un certo interesse.

La rigorizzazione della teoria delle equazioni algebriche in un corpo finito, portata avanti da Cipolla e dalla sua "scuola", fu motivata principalmente dall'esigenza di ricondurre la risoluzione delle equazioni a metodi canonici, rendendo obsoleti quelli che richiedevano la determinazione per tentativi di alcuni elementi necessari per trovarne le radici.

Per realizzare ciò, come s'è visto, nuovi e importanti concetti vennero introdotti come, per esempio, da Cipolla quello di *sistema completo di n -esimo grado (mod p)* e di *funzione indicatrice di un sistema di numeri in un corpo finito*; il concetto, ad opera di Mignosi, di *sistema completo di grado n di numeri di un corpo finito*; per non parlare dei metodi introdotti da Sansone per trattare in modo diretto la risoluzione delle congruenze di terzo e di quarto grado. Ciò comportò, da parte dei matematici italiani, non solo lo

studio approfondito dei corpi finiti, ma anche un progresso decisivo per gli studi propriamente algebrici. In effetti, le ricerche di Cipolla e dei suoi continuatori costituirono un insieme isolato di lavori rispetto ai campi di ricerca che interessarono i matematici italiani nel periodo tra le due guerre mondiali. Infatti, la maggior parte delle ricerche matematiche di quel periodo riguardavano la geometria algebrica, l'analisi reale e la fisica matematica. In geometria algebrica erano universalmente noti i risultati fondamentali di G. Castelnuovo (1865-1952), F. Enriques (1871-1946), G. Fano (1871-1952) e F. Severi (1879-1961) sulla teoria delle curve sopra una superficie, sulla classificazione delle superfici e sulla teoria generale delle varietà algebriche. In analisi reale risultati importanti furono raggiunti con i lavori di V. Volterra (1860-1940), di G. Fubini (1879-1943), di E. E. Levi (1883-1917) e di L. Tonelli (1885-1946). In fisica matematica si ebbero risultati significativi da parte di Cesàro, Bianchi, Ricci Curbastro (1853-1921) e T. Levi-Civita (1873-1941) [Bottazzini, 1981], [Guerraggio e Nastasi, 1993]. Le ricerche di aritmetica superiore riguardavano soprattutto la teoria analitica dei numeri e la teoria degli ideali, trattata specialmente da Bianchi.

Ciò nonostante, l'opera di Cipolla e della sua scuola riuscì a mantenere vivo l'interesse per le ricerche nella teoria algebrica dei numeri, contribuendo, insieme all'opera contemporanea di Bianchi a Pisa, relativa alla teoria degli ideali, alla crescita di un maggiore interesse per l'algebra, che era una disciplina coltivata solo da pochi matematici [Brigaglia, 1987], e soprattutto a sottrarre la teoria dei numeri al ruolo di cenerentola della ricerca matematica.

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A Note on the existence and uniqueness of solutions to a free piston problem

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Key words: Piston problem, Free boundary value problems, Partial differential equations.

Abstract - A free boundary value problem related to the motion of a piston in a viscous gas is considered. A uniqueness theorem is proved and a contribution to the global in time existence is given. Moreover, an example with periodic boundary conditions is studied by numerical simulations.

Riassunto - Si considera un problema di frontiera libera relativo al moto di un pistone in un gas viscoso. Si prova un teorema di unicità e viene apportato un contributo al problema dell'esistenza globale. Inoltre, con simulazioni numeriche si studia un esempio con condizioni al bordo periodiche.

1 - INTRODUCTION

In a previous paper a free boundary value problem has been discussed with reference to the motion of a piston in a cylinder filled with gas. The main result was an existence theorem, local in time [2].

The purpose of the present paper is to obtain a uniqueness theorem for the same problem. Moreover, we also give a contribution to the global (in time) existence under admissible boundary conditions.

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The plan of the paper is as follows. In Section 2 we recall the mathematical model concerning the one-dimensional linearized motion of a viscous isentropic gas contained in a cylinder bounded by a fixed wall and a free piston. Section 3 is dedicated to an auxiliary problem. Precisely, we assume that the piston motion is known and we obtain a uniqueness theorem by using a useful inequality given in [8]. Successively, in Section 4, we recall the existence theorem given in [2] for the problem considered in Sect. 3. Section 5 is dedicated to the existence and uniqueness of the complete free piston problem. In particular we put in evidence a boundary condition which guarantees the global (in time) existence. Finally, in Section 6 an example of periodic (in time) boundary data is considered and a numerical simulation is furnished.

2 - FREE PISTON PROBLEM IN A VISCOUS GAS

We briefly recall the mathematical model we have considered in [2], in order to examine a special situation of the motion of a piston in a cylinder filled with gas. Let $x = 0$ be the fixed end of the cylinder, $x = s(t)$ the location of the piston at time t , $u(x, t)$ the gas velocity, $\sigma(x, t)$ the condensation and $F(s, \dot{s}, t)$ the force acting directly on piston. We assume the gas is viscous and we consider the linear motions, so that the resulting nonlinear free boundary value problem is described by the following equations

$$(2.1) \quad u_t + c^2 \sigma_x = \varepsilon u_{xx}, \quad 0 < t \leq T, \quad 0 < x < s(t),$$

$$(2.2) \quad \sigma_t + u_x = 0, \quad 0 < t \leq T, \quad 0 < x < s(t),$$

$$(2.3) \quad u(x, 0) = u_0(x), \quad \sigma(x, 0) = \sigma_0(x), \quad 0 < x < a,$$

$$(2.4) \quad u(0, t) = g(t), \quad 0 < t < T,$$

$$(2.5) \quad u(s(t), t) = \dot{s}(t), \quad 0 < t < T,$$

$$(2.6) \quad \ddot{s} = \tilde{\alpha}[c^2 \sigma(s(t), t) - \varepsilon u_x(s(t), t)] + F(\dot{s}, s, t), \quad \tilde{\alpha} = \text{const.}, \quad 0 < t < T,$$

$$(2.7) \quad \dot{s}(0) = b, \quad s(0) = a > 0.$$

We assume that $F(\dot{s}, s, t)$ is a bounded continuous function which satisfies a Lipschitz condition with respect to $\dot{s}, s$

$$(2.8) \quad |F(\dot{s}, s, t)| \leq C_F, \quad |F(\dot{s}_1, s_1, t) - F(\dot{s}_2, s_2, t)| \leq L\{|\dot{s}_1 - \dot{s}_2| + |s_1 - s_2|\}.$$

Furthermore,

$$(2.9) \quad u_0, \sigma_0 \in C^2([0, a]), \quad g \in C^1([0, T]),$$

and

$$(2.10) \quad g(0) = u_0(0), \quad \dot{s}(0) = u_0(a).$$

Then, with reference to problem (2.1)-(2.7), an existence theorem has been proved in [2], under the above hypotheses.

3 - AN AUXILIARY PROBLEM. UNIQUENESS

We first discuss the uniqueness of the problem

$$(3.1) \quad u_t + c^2 \sigma_x - \varepsilon u_{xx} = 0, \quad 0 < t < T, \quad 0 < x < s(t),$$

$$(3.2) \quad \sigma_t + u_x = 0, \quad 0 < t < T, \quad 0 < x < s(t),$$

$$(3.3) \quad u(x, 0) = u_0(x), \quad \sigma(x, 0) = \sigma_0(x), \quad 0 < x < a = s(0),$$

$$(3.4) \quad u(0, t) = g(t), \quad u(s(t), t) = h(t), \quad 0 < t < T,$$

where $x = s(t)$ is a *prefixed* curve and

$$(3.5) \quad \Delta = \inf_{0 \leq t \leq T} s(t) > 0.$$

We assume that the function $x = s(t)$ is such that

$$(3.6) \quad s \in C^1([0, T]), \quad |\dot{s}(t)| \leq N, \quad N = \text{constant} > 0.$$

We denote by (u, σ) the solution of (3.1), (3.2) with vanishing initial and boundary conditions

$$(3.7) \quad u_0(x) = \sigma_0(x) = 0, \quad 0 < x < a, \quad g(t) = h(t) = 0, \quad 0 < t < T.$$

In order to show that (u, σ) is unique, we also consider the solution (u_1, σ_1) of (3.1), (3.2) such that

$$(3.8) \quad u_0(x) = \sigma_0(x) = \eta, \quad 0 < x < a, \quad g(t) = h(t) = \eta, \quad 0 < t < T.$$

Consequently, the functions

$$(3.9) \quad u_2 = u + u_1, \quad \sigma_2 = \sigma + \sigma_1,$$

satisfy (3.1), (3.2) with initial and boundary conditions (3.8).

Now, with reference to (u, σ) , we introduce the weighted Liapunov functional V defined by

$$(3.10) \quad V(t) = (1/2) \int_0^{s(t)} [s(t) - x]^p [u^2(x, t) + c^2 \sigma^2(x, t)] dx, \quad 0 \leq t \leq T,$$

where $p = \text{const.} > 1$. Obviously, because of (3.9),

$$(3.11) \quad V(t) \leq 2[V_1(t) + V_2(t)],$$

where

$$V_i(t) = (1/2) \int_0^{s(t)} [s(t) - x]^p [u_i^2(x, t) + c^2 \sigma_i^2(x, t)] dx, \quad 0 \leq t \leq T.$$

Along the solutions (u_i, σ_i) , it turns out that

$$(3.12) \quad \begin{aligned} \dot{V}_i &= (ps/2) \int_0^s (s-x)^{p-1} (u_i^2 + c^2 \sigma_i^2) dx + \int_0^s (s-x)^p (u_i u_{it} + c^2 \sigma_i \sigma_{it}) dx; \\ \dot{V}_i &= (ps/2) \int_0^s (s-x)^{p-1} (u_i^2 + c^2 \sigma_i^2) dx - \varepsilon \int_0^s (s-x)^p u_{ix}^2 dx + \\ &\quad + (\varepsilon/2)p(p-1) \int_0^s (s-x)^{p-2} u_i^2 dx - pc^2 \int_0^s (s-x)^{p-1} u_i \sigma_i dx + \gamma_i(t), \end{aligned}$$

where,

$$(3.13) \quad \gamma_i(t) = -\eta s^p(t) [\varepsilon u_{ix}(0, t) - c^2 \sigma_i(0, t)] - \eta^2 (\varepsilon/2) p s^{p-1}(t).$$

Since, it is too

$$(3.14) \quad c^2 |u_i \sigma_i| \leq c^2 (u_i^2 + \sigma_i^2)/2 \leq (1 + c^2)(u_i^2 + c^2 \sigma_i^2)/2,$$

from (3.12) it follows

$$(3.15) \quad \begin{aligned} \dot{V}_i(t) &\leq (p/2)(N+1+c^2) \int_0^s (s-x)^{p-1} (u_i^2 + c^2 \sigma_i^2) dx + \\ &\quad + (p/2)\varepsilon(p-1) \int_0^s (s-x)^{p-2} (u_i^2 + c^2 \sigma_i^2) dx + \gamma_i(t). \end{aligned}$$

Now, with reference to the functions $u_i^2 + c^2 \sigma_i^2$, we can apply a basic result (Theorems 1 and 2) proved in [8]

$$(3.16) \quad \int_0^s (s-x)^{p-2} (u_i^2 + c^2 \sigma_i^2) dx \leq \alpha_i(t) \int_0^s (s-x)^{p-1} (u_i^2 + c^2 \sigma_i^2) dx,$$

where $\alpha_i(t) \in L^2([0, T])$ is positive and independent of p . Thus, from (3.15) it follows

$$(3.17) \quad \dot{V}_i \leq (p/2)[N+1+c^2+\varepsilon(p-1)\alpha_i(t)] \int_0^s (s-x)^{p-1} (u_i^2 + c^2 \sigma_i^2) dx + \gamma_i.$$

Hence,

$$(3.18) \quad \dot{V}_i(t) \leq k_i(t)V_i(t) + \gamma_i(t),$$

where

$$(3.19) \quad k_i(t) = (p/2)[N + 1 + c^2 + \varepsilon(p-1)\alpha_i(t)]\alpha_i(t).$$

Finally, by setting

$$(3.20) \quad c_i(t) = \int_0^t k(\tau) d\tau,$$

from (3.18) it easily follows

$$(3.21) \quad V_i(t) \leq V_i(0) \exp[c_i(t)] + \int_0^t \gamma_i(\tau) \exp[c_i(t) - c_i(\tau)] d\tau, \quad t \in [0, T].$$

Now, we note that

$$(3.22) \quad \lim_{\eta \rightarrow 0} V_i(t) = 0,$$

since $V_i(0)$ is vanishing when $\eta \rightarrow 0$ and so does $\gamma_i(t)$. If we consider (3.22) and (3.11) and observe that $V(t)$ is independent of η , we can conclude that $V(t) = 0$, i.e. the zero solution is unique.

So, we can state

Theorem 3.1. *Under the hypothesis (3.6), the problem (3.1)-(3.5) has a unique solution in the class of the smooth solutions such that $u(\varepsilon u_x - c^2 \sigma)$ is bounded for $x = 0$ and $x = s(t)$.*

4 - EXISTENCE

An existence theorem for (3.1)-(3.4) has been obtained in [2], by assuming that, besides (3.5), the following hypotheses are satisfied

$$(4.1) \quad g, h, s \in C^1([0, T]),$$

$$(4.2) \quad u_0, \sigma_0 \in C^2([0, a]),$$

together with compatibility conditions

$$(4.3) \quad g(0) = u_0(0), \quad h(0) = u_0(a).$$

The proof is based on the fundamental solution $K(x, t)$ of the operator

$$(4.4) \quad (\varepsilon \partial_t + c^2) \partial_{xx} - \partial_{tt}.$$

The function K , explicitly determined in [3], is never negative and has properties similar to the ones of the heat fundamental solution.

More precisely, if we define

$$(4.5) \quad E(\dot{s}(t); x, t) = \dot{s}(t)K_t(x, t) + (\varepsilon\partial_t + c^2)K_x(x, t),$$

then, a solution of (3.1)-(3.4) can be given by means of two auxiliary functions φ and ψ

$$(4.6) \quad u(x, t) = v(x, t) + 2 \int_0^t \varphi(\tau)(\varepsilon\partial_t + c^2)K_x(x, t - \tau)d\tau + \\ + 2 \int_0^t \psi(\tau)E(\dot{s}(\tau); x - s(\tau), t - \tau)d\tau,$$

$$(4.7) \quad \sigma(x, t) = z(x, t) - 2 \int_0^t \dot{\varphi}(\tau)K(x, t - \tau)d\tau - \\ - 2 \int_0^t \dot{\psi}(\tau)K(x - s(\tau), t - \tau)d\tau,$$

where the functions φ, ψ are uniquely determined by the boundary data (3.4). Furthermore,

$$(4.8) \quad v(x, t) = \int_{\mathbb{R}} u_0(\xi)K_t(x - \xi, t)d\xi - \int_{\mathbb{R}} c^2\sigma'_0(\xi)K(x - \xi, t)d\xi,$$

$$(4.9) \quad z(x, t) = \int_{\mathbb{R}} \sigma_0(\xi)K_t(x - \xi, t)d\xi - \int_{\mathbb{R}} (u'_0 + \varepsilon\sigma''_0)(\xi)K(x - \xi, t)d\xi,$$

after a smooth extension with compact support of the initial data.

Since, as it easy to verify the solution (4.6), (4.7) verifies all the assumptions of uniqueness theorem 3.1, we conclude it is unique.

5 - EXISTENCE AND UNIQUENESS OF THE FREE BOUNDARY VALUE PROBLEM

Let us consider the free boundary value problem (2.1)-(2.7). By taking into account the results of Sections 3 and 4, it follows that the unique solution of problem (2.1)-(2.5) is given by (4.6)-(4.7) where φ and ψ satisfy the following integral equations

$$(5.1) \quad \varphi(t) = v(0, t) - g(t) + 2 \int_0^t \psi(\tau)E(\dot{s}(\tau); -s(\tau), t - \tau)d\tau,$$

$$(5.2) \quad \psi(t) = \dot{s}(t) - v(s(t), t) - 2 \int_0^t \varphi(\tau)(\varepsilon\partial_t + c^2)K_x(s(t), t - \tau)d\tau -$$

$$-2 \int_0^t \psi(\tau) E(\dot{s}(\tau); s(t) - s(\tau), t - \tau) d\tau,$$

$$(5.3) \quad \dot{\varphi}(t) = v_t(0, t) - \dot{g}(t) + 2 \int_0^t \dot{\psi}(\tau) (\varepsilon \partial_t + c^2) K_x(-s(\tau), t - \tau) d\tau,$$

$$(5.4) \quad \dot{\psi}(t) = \ddot{s}(t) - \dot{v}(s(t), t) - 2 \int_0^t \dot{\varphi}(\tau) E(\dot{s}(\tau); s(t), t - \tau) d\tau - \\ - 2 \int_0^t \dot{\psi}(\tau) E(\dot{s}(\tau); s(t) - s(\tau), t - \tau) d\tau.$$

When $s(t)$ is given, equations (5.1)-(5.4) are enough to close the problem. In our situation $s(t)$ is unknown and one more equation is necessary. But, from (2.6) we can deduce

$$(5.5) \quad \ddot{s}(t) = F(\dot{s}, s, t) + \tilde{\alpha} \int_{\mathbb{R}} [c^2 \sigma_0(\xi) \partial_t - u'_0(\xi) (\varepsilon \partial_t + c^2)] K(s(t) - \xi, t) d\xi - \\ - 2\tilde{\alpha} \int_0^t [\dot{\varphi}(\tau) (\varepsilon \partial_t + c^2) K(s(t), t - \tau) + \dot{\psi}(\tau) (\varepsilon \partial_t + c^2) K(s(t) - s(\tau), t - \tau)] d\tau.$$

Under the hypotheses (2.8)-(2.10), by using estimates of the fundamental solution K and its derivatives [3, 2], it can be proved that Volterra system (5.1)-(5.5) has a unique continuous solution [2]. Consequently, $u(x, t)$, $\sigma(x, t)$ and $s(t)$ are unique and we can state

Theorem 5.2. *Under the hypotheses (2.8)-(2.10) there exist a unique smooth solution (u, σ) and a unique $s \in C^2([0, T])$ for the free boundary value problem (2.1)-(2.7).*

The theorem holds until $s(t)$ is greater or equal a strictly positive constant, since the solution of the free boundary value problem is deduced from the one of the auxiliary problem (3.1)-(3.4) based on the assumption (3.5):

$$(5.6) \quad \Delta = \inf_{0 \leq t \leq T} s(t) > 0.$$

However, it is not difficult to find a condition under which (5.6) is fulfilled for the free boundary value problem (2.1)-(2.7). Indeed, if we assume $u(0, t) = g(t) > 0$, then the existence of $\bar{t} > 0$ such that

$$\lim_{t \rightarrow \bar{t}} s(t) = s(\bar{t}) = 0, \quad \dot{s}(\bar{t}) \leq 0,$$

leads to a contradiction, as $g(\bar{t}) = u(0, \bar{t}) = u(s(\bar{t}), \bar{t}) = \dot{s}(\bar{t}) \leq 0$.

Moreover, in the next section we give a numerical example of a solution that exists for large times under more general boundary conditions.

6 - A NUMERICAL SIMULATION

Let us consider the free boundary value problem (2.1)-(2.7) with constant initial data and periodic boundary conditions

$$(6.1) \quad u_0(x) = 0, \quad \sigma(x) = \bar{\sigma}_0 = \text{const.}, \quad x \in [0, s(0)], \quad \dot{s}(0) = 0.$$

$$(6.2) \quad g(t) = u(0, t) = G \sin t, \quad G = \text{const.}, \quad t > 0.$$

Our aim is to furnish a numerical solution by using the finite difference method. The simulation of this section is based on a suitable program written in C language.

Since, as it has been noted, the function u satisfies the third order operator (4.4), we use the following scheme

$$(6.3) \quad \begin{aligned} u_{ij+1} = & 2(1 - \rho^2 - r)u_{ij} - (1 - 2r)u_{ij-1} + \\ & + (\rho^2 + r)(u_{i+1j} + u_{i-1j}) - r(u_{i+1j-1} + u_{i-1j-1}), \end{aligned}$$

where $u_{ij} = u(x_i, t_j) = u(i \Delta x, j \Delta t)$, and

$$\rho = c \Delta t / \Delta x, \quad r = \varepsilon \Delta t / \Delta x^2,$$

satisfy the stability condition [6]

$$\rho^2 + 2r \leq 1.$$

Next, on each time step, we use

$$(6.4) \quad \sigma_{ij+1} = \sigma_{ij} - (\Delta t / 2 \Delta x)(u_{i+1j} - u_{i-1j})$$

to compute $\sigma_{ij} = \sigma(i \Delta x, j \Delta t)$. Moreover, from (2.5), (2.6) we find the free boundary

$$(6.5) \quad s_{j+1} = s_j + \Delta t u(s_j, t_j), \quad s_j = s(t_j),$$

and the value $u(s_j, t_j)$ on the free boundary

$$(6.6) \quad u(s_{j+1}, t_{j+1}) = u(s_j, t_j) + \Delta t \{ \tilde{\alpha} [c^2 \sigma(s_j, t_j) - \varepsilon u_x(s_j, t_j)] + F_j \},$$

where

$$F_j = F(s_j, \dot{s}_j, t_j).$$

The behaviour of the free boundary $x = s(t)$ obtained when

$$c = \varepsilon = \tilde{\alpha} = 1, \bar{\sigma}_0 = .1, G = 50, s_0 = 40, F = 0, t \in [0, 1000],$$

is sketched in Fig. 1.

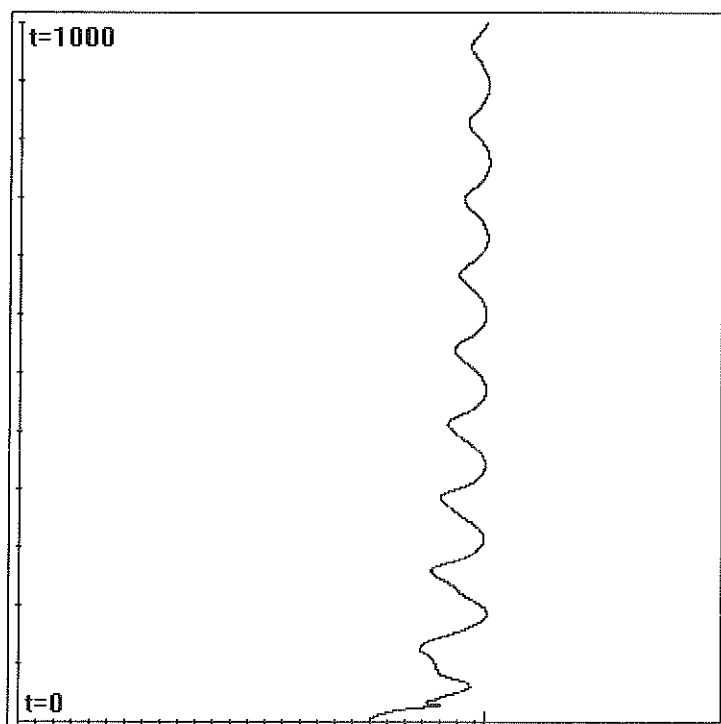


Fig. 1

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Sulla DHI-melanina

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Key words: melanins, fullerenes, cosmoids, MALDI, conducting polymers, X-ray diffraction, DHI-melanin, black particles, tensegrity.

Abstract – A 16 indole unit protoparticle for the DHI-melanin is suggested. The porphyrin-like structure gives account of some melanin's properties.

Riassunto – Una protoparticella del nero di DHI viene proposta. La presenza di un sito porfirino-simile nella struttura riferita a 16 unità indoliche rende conto di alcune proprietà delle melanine.

Le particelle nere sono presenti nella biosfera (melanine), nella litosfera (minerali, carbone, petrolio, grafite, scisti bituminosi, fullereni, solfuri), nell'atmosfera (fumi, inquinanti primari e secondari), nell'idrosfera (mare, laghi, fiumi) e nel cosmo (cosmoidi e grafite). Le particelle degli spazi interstellari possono essere dei neri derivati da uno o più sistemi eterociclici come mostrato nella tavola 1.

Materiali neri si possono ottenere anche per sintesi come ad esempio il nero di pirrolo. Tutti questi materiali hanno in comune un caratteristico segnale EPR (*Electronic Paramagnetic Resonance*) e la conducibilità in semiconduttori amorfi con piccolo gap.

Le melanine [1] sono pigmenti prodotti da cellule specializzate tipo melanociti nel caso dei vertebrati omeoterme e melanofori nel caso di quelli eteroterme. Esempi di melanine sono quelle dell'iride, dei peli, della cute;

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particolari sono quelle del cervello (*substantia nigra*) e dell'inchiostro dei cefalopodi.

Le melanine costituiscono un materiale insolubile ed infusibile difficilmente purificabile. Per questo motivo quasi tutto il materiale sperimentale raccolto nello studio chimico e fisico delle melanine è di difficile interpretazione.

Le melanine possono avere origine da diverse molecole dette melanogeni. Questi sono in genere composti ossidrilati (ortodifenolici) di sistemi aromatici quali benzene, indolo, pirrolo, piridina, chinolina. Sostanze di interesse biologico che hanno la proprietà di produrre particelle nere sono la DOPA, il DHI (5,6-dihydroxindole), il DHICA (5,6-dihydroxindole-2-carboxylic acid), la dopamina, l'adrenalina, la serotonina, la 5,6-diidrossitriptamina [2], la 6,7-diidrossi-1-metil-tetraidroisochinolina (salsolinolo) [3]. Tutte le sostanze riportate nella tavola 1 possono, in diverse condizioni, produrre materiale nero con una reazione complessa di tipo radicalico. Le particelle (in biologia melanosomi) si presentano in genere sotto forma ovoidale o sferica (rugby or soccer ball). E' possibile che questi materiali costituiscano dei sistemi a gabbia di tipo fullerenico o strutture grafittiche a sandwich.

La prima fase polimerica della melanogenesi consiste nella formazione di oligomeri in cui sono presenti catene policoniugate dette anche spine di Little [4]. Si deve osservare che la spina può da sola risultare anche isolante. Se opportunamente sostituita da gruppi quaternari laterali in cui la carica può interagire con gli elettroni che si muovono nella spina, si produce lo stato superconduttivo e cioè si forma una particolare macromolecola organica dotata delle proprietà di un superconduttore alla luce della teoria BCS [5]. E' curioso notare che nel modello Little la spina era il nero di acetilene e le catene laterali erano costituite dallo ioduro di dietilcianina, un noto fotosensibilizzatore. Considerando che il polimero derivato dal DHI si autoassembli a partire da 16 unità (magic number?) si avrebbero le strutture (aromatiche) rappresentate nelle tavole 2 e 3. Le strutture, tra le tante possibili, delle tavole 2 e 3 differiscono tra loro per la presenza di anelli furanici. Queste sono teoricamente dei semiconduttori [7] o dei superconduttori organici [5].

La seconda fase di polimerizzazione è caratterizzata dall'autoassemblaggio delle varie unità fino a raggiungere strutture grafittiche del tipo mostrato nella tavola 4.

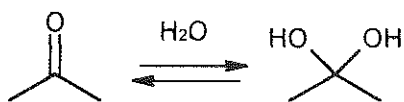
La prima fase si può verificare esaminando i prodotti della reazione con la spettrometria di massa MALDI e MALDI-TOF. Si può rilevare la presenza di oligomeri (indolici) fino ad 11 unità ossidando la dopamina con perossidasi/H₂O₂ e fino ad 8 ossidando la tirosina con tirosinasi [2], [6].

La struttura mostrata nella tavola 4, derivata dalla polimerizzazione e policondensazione del DHI, può rappresentare una delle possibili protoparticelle e spiegare alcune proprietà chimiche e fisiche delle melanine e delle particelle nere, suggerendo subito l'esistenza di bande di energia di elettroni liberi e delle

proprietà di semiconduttore [7]. Da un punto di vista chimico va tenuto presente che la particella dovrebbe essere rappresentata, secondo la teoria poliindolchिनonica [1], da $C_8H_3O_2N$, invece essa tende ad una formula avente le proporzioni atomiche $C_8H_{5.6}O_{3.7}N_{1.1}$. Altra discrepanza è il fatto che la particella possiede gruppi $-OH$, $-COOH$ e centri cationici capaci di legare acidi come HCl .

Tutte le melanine si trovano sotto forma di sali insolubili. Per esempio la melanina dell'inchiostro della *Sepia officinalis* è un sale di Ca e Mg . Gli acidi melanici corrispondenti perdono CO_2 per riscaldamento, anche blando. Le melanine sono sensibili alla luce, alla pressione, all'ossigeno e ai perossidi. Tutte queste variabili rendono difficile lo studio delle melanine. Le melanine e la DHI-melanina reagiscono con Cl_2 , CH_2N_2 , sali di diazonio, si legano con H_2O , droghe, ioni e gas mostrando le stesse proprietà del carbone attivo o charcoal. La gabbia melaninica esplode sotto l'azione del laser [8] ricordando il comportamento dei fullereni giganti. Il laser del MALDI produce la frammentazione della sepiomelanina con formazione di numerosi prodotti a basso peso molecolare (da essere pubblicato). La chimica e fisica dei fullereni giganti (es. C_{540}) è interessante per lo studio delle melanine. Infatti un fullerene gigante è essenzialmente un monostrato di esagoni con carboni sp^2 che si piega tra dodici cuspidi pentagonali necessarie per la chiusura. Queste strutture hanno fornito la chiave per interpretare l'interno delle microparticelle di carbonio [9].

Le sostanze e gli ioni si legano alla melanina sia per una reazione chimica di salificazione (carbossili, basi azotate) o con complessi di coordinazione tipo porfirine, sia per un fenomeno interstiziale. Questi fenomeni sono da tempo conosciuti nei carboni attivi. Composti interstiziali si possono formare nelle strutture proposte anche con i gas. Recentemente [10] sono stati preparati composti interstiziali da piccoli fullereni C_{60} e C_{70} con Argo, Cripto e Xeno di formula ArC_{70} , KrC_{70} e XeC_{70} . I derivati con Xeno e Cripto possono essere stabili a temperatura ambiente anche per mesi. Per le melanine l'intrappolamento di O_2 ed H_2O può suggerire nuovi meccanismi e funzioni biologiche. Le melanine devono aver avuto un importante ruolo nell'era pre-enzimatica. A questo punto va ricordato che l'idratazione e la disidratazione delle melanine può essere imputabile ad una reazione di addizione di H_2O ad un sistema carbonilico (chinonico) già descritto in letteratura [11].



Idratazione di gruppi chinonici

I granuli di melanina (DHI-melanina) sono capaci di legarsi a diverse sostanze chimiche le quali possono essere accumulate nel tempo [12], [13].

Oltre che per numerose sostanze organiche, notevole è anche l'affinità per ioni inorganici [14], [15]. Tutte queste proprietà sono spiegabili con la presenza di un sito porfirino-simile rappresentato nelle tavole 2, 3 e 5. Le strutture indicate sono inoltre in accordo con la formazione dell'acido pirrolettetracarbossilico nell'ossidazione delle eumelanine. Le melanine possono essere viste anche come importanti pedine nell'autoassemblaggio che va da: atomi → molecole → cellule → tessuti. E' stato suggerito che l'autoassemblaggio di atomi di carbonio, molecole di H_2O , proteine, virus, cellule segue un principio architettonico detto tensegrità [15], [16]. La melanina è un materiale geodetico universale sensibile alla luce e all'ossigeno. Tale materiale è capace di catalizzare, iniziare, influire su numerose reazioni chimiche, comprese quelle all'origine delle prime molecole dell'autoassemblaggio. Si può inoltre supporre che l'evoluzione biologica potrebbe essere iniziata su un film di melanina piuttosto che aver avuto origine dalla cosiddetta zuppa biologica.

Le strutture porfiriniche proposte nelle tavole 2, 3, 4 e 5 si possono assemblare in sandwich grafitici (tavola 4) con interspazi di 3.4 Å in accordo con il valore trovato mediante diffrazione dei raggi X per molte melanine [18]. Nel sistema porfirino-simile la distanza tra gli atomi di azoto è 2.7 Å, mentre nella porfina è di 4.3 Å. Il sistema porfirinico permette la formazione di complessi di grande interesse in biologia e può spiegare l'affinità della melanina per metalli e ioni, l'attività perossidasi, l'assorbimento di gas, la coordinazione di molecole di H_2O , la conduttività. Naturalmente le strutture porfirino-simili rappresentano solo una parte della complessa ed amorfa particella. E' comunque da notare che centri porfirinici compaiono anche in modelli con unità indoliche legate random in accordo con i dati sperimentali di diffrazione dei raggi X della tirosino-melanina [18]. Le melanine sono rappresentate da particelle di dimensione $R \sim 15$ Å. Strutture planari disordinate di 4-8 unità diossindoliche sono assemblate in sandwich grafitici a 4-5 strati e con interspazi di 3.45 Å [18], [19]. La presenza di sistemi porfirino-simili suggerisce la possibilità che si realizzi la superconduttività attraverso complessi metallici o con la sola partecipazione di elettroni π dei macrocicli porfirinici sovrapposti [20]. Va inoltre ricordato che molti complessi porfirinici, specie del ferro [4], [21], hanno colori neri elettrici (nero-blu, nero-oro, nero-rame) tipici dei semiconduttori inorganici essendo essi stessi da considerare dei semiconduttori o superconduttori.

La conduttività delle melanine, le gabbie fullereniche, le proprietà di superficie, gli elettroni spaiati, l'idratazione, l'attività interstiziale, cambiano reversibilmente applicando un potenziale elettrico. Tali proprietà rendono questi semiconduttori amorfi naturali di grande interesse per la ricerca biologica. La melanina elettroattiva rappresenta un tipo di substrato colturale che potrebbe giocare un ruolo nel controllo della forma e della funzione di aderenza cellulare.

APPENDICE

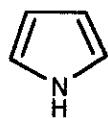
Le strutture riportate nelle tavole 1, 2, 3 e 4 sono state realizzate su PC con il programma Isis Draw versione 2.1.1 della MDL Information Systems inc.

Il modello molecolare 3D 'ball & stick' riportato nella tavola 5 è stato realizzato con il programma WebLab versione 3.0 della Molecular Simulations inc. ed elaborato graficamente con Rasmol versione 2.6.

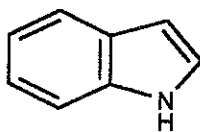
Per ulteriori informazioni riguardanti lo studio delle melanine e dei modelli molecolari si rimanda ai seguenti indirizzi Internet:

- Melanine - <http://www.tightrope.it/nicolaus>
- Modellistica molecolare - <http://www.dichi.unina.it/multimedia/biomolecole/>

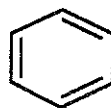
Tavola 1



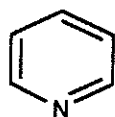
Pirroto



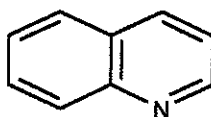
Indolo



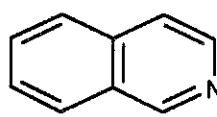
Benzene



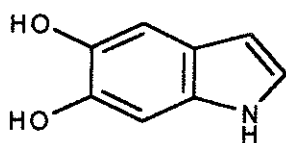
Piridina



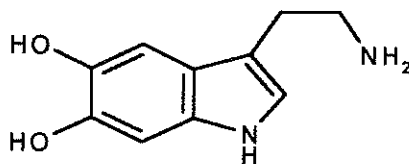
Chinolina



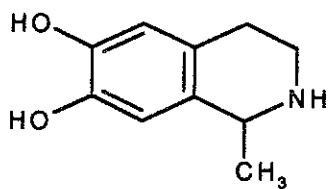
Isochinolina



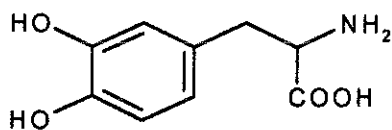
DHI
5,6-diidrossindolo



DHT
5,6-diidrossitriptam ina



Salsolinolo



Dopa

Tavola 2

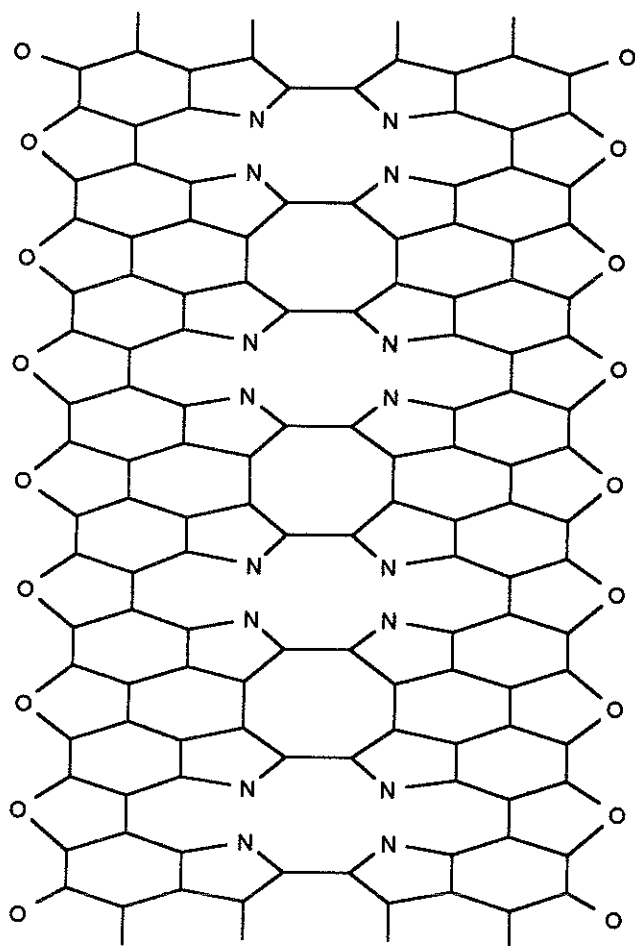


Tavola 3

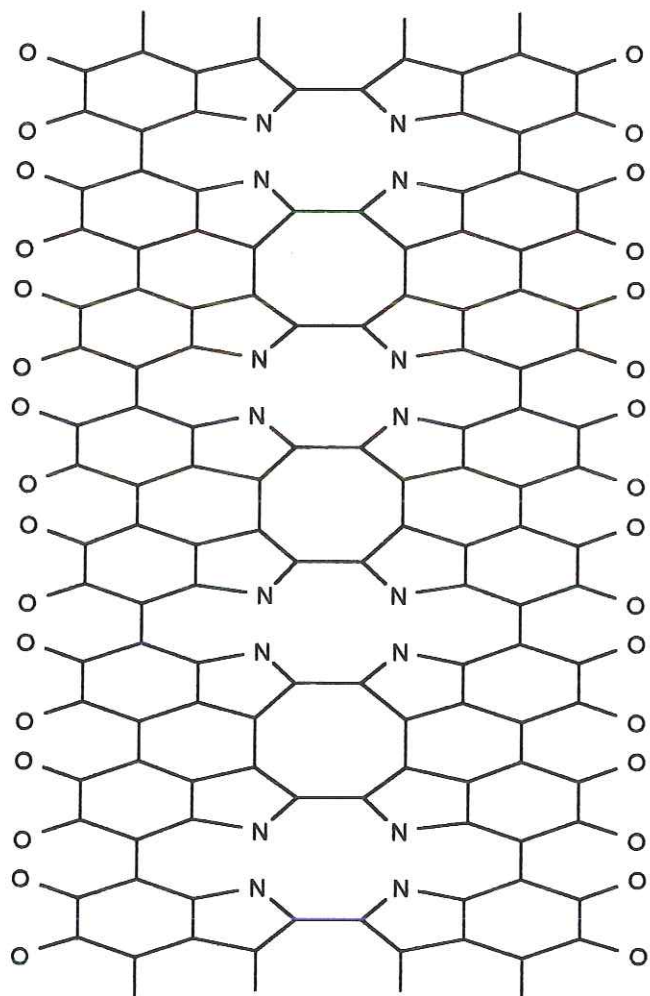


Tavola 4

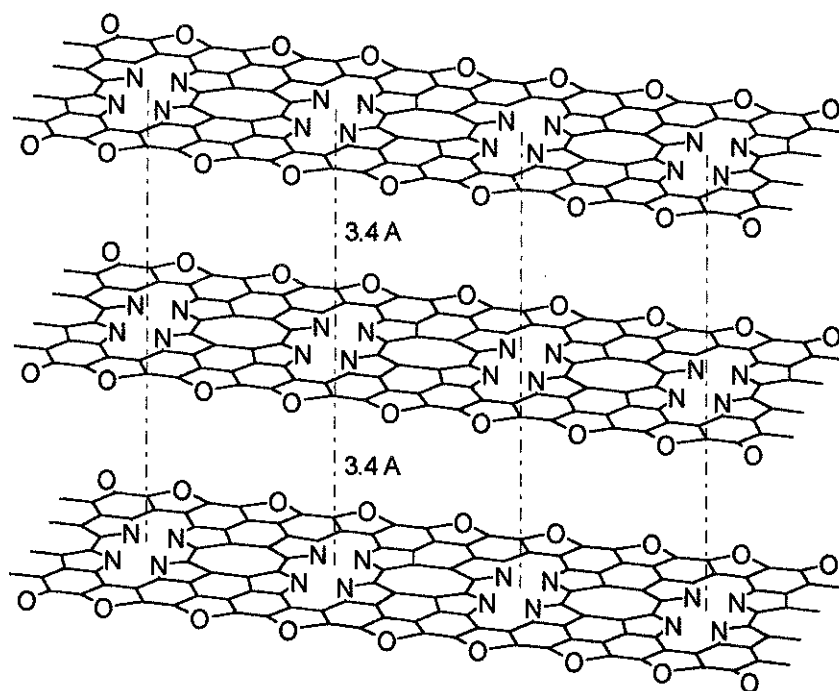
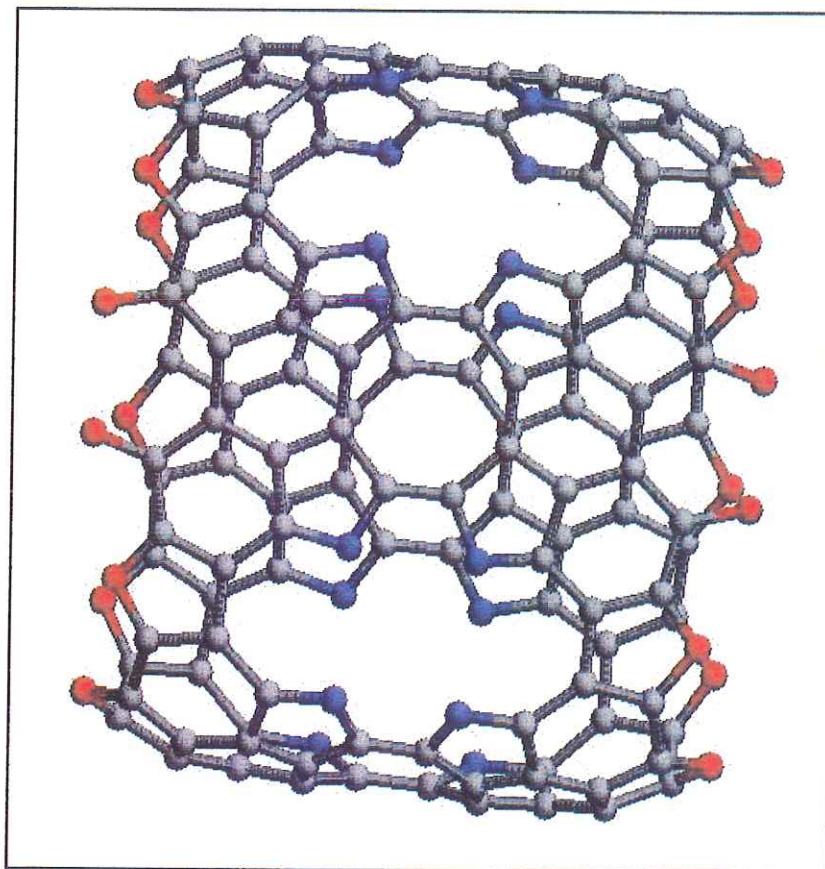


Tavola 5



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AN ALGEBRAIC APPROACH TO AN AUTONOMOUSLY ADAPTABLE SYSTEM

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Abstract. Let us consider the unbounded sequence S , generated one symbol at the time, the symbols being the elements of a finite alphabet I . We call *example* a triplet of not necessarily consecutive symbols, belonging to S and satisfying given conditions. We consider the subset A of the examples such that if $E \in A$ there exists at least one finite subsequence of S in which the repetition cadence of E satisfies given conditions. By definition, there is a one-to-one correspondence between the A elements and their first symbols, ie, if the triplet E belongs to A , then, its first symbol identifies the example E and vice versa. Let $\mathfrak{K}$ be the first symbol of the set A generic element. Then, we present a formal system, defined on the sequence S symbols, automatically projecting any instance of $\mathfrak{K}$, as soon as it becomes an S element, into the concatenation of the projections of the very same example second and third symbols, even if the latter two symbols do not appear in S anymore.

Riassunto. Consideriamo una sequenza illimitata S , generata un simbolo alla volta, i simboli essendo gli elementi di un alfabeto finito I . Definiamo *esempio* una tripla di simboli appartenenti ad S e soddisfacenti condizioni date. Chiamiamo A il sottoinsieme degli esempi tali che se $E \in A$ esiste almeno una sottosequenza finita di S nella quale la cadenza di ripetizione di E soddisfa condizioni date. Per definizione, esiste una corrispondenza biunivoca senza eccezioni tra gli elementi di A ed i loro primi simboli, cioè se la tripla E appartenente ad A allora il suo primo simbolo individua E e vice versa. Sia $\mathfrak{K}$ il primo simbolo del generico elemento di A , nel lavoro si presenta un sistema formale, definito sui simboli appartenenti ad S , che automaticamente proietta le istanze di $\mathfrak{K}$, presenti in S , nella concatenazione delle proiezioni del secondo e terzo simbolo appartenenti alla tripla individuata da $\mathfrak{K}$, anche se gli ultimi due simboli non sono più generati come elementi di S .

1. INTRODUCTION.

Let us call S an unbounded sequence S , generated one symbol after the other, each symbol being drawn from the I symbols alphabet I $i_1, i_2, \dots, i_j, \dots, i_I$. Let the $0 < P < I$ symbols $i_1, i_2, \dots, i_P$ belong to P , a proper subset of I .

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We are going to define A , a proper subset of I , relative to S : initially the set A is empty. Let us call $a_1, a_2, \dots, a_A$ the A elements, with $A < I$. As we shall see, by definition A and P have no common element, ie, they are disjoint subsets of I relative to any given sequence S of unbounded length.

We call S the sequence of unbounded length: $i_{j_1}, i_{j_2}, \dots, i_{j_q}, \dots, i_{j_m}, \dots, i_{j_n}, \dots$, where $i_{j_1}, i_{j_2}, i_{j_q}, i_{j_m}, i_{j_n}$ represent symbols belonging to I . In S the symbol represented by i_{j_q} is different from the symbol represented by either i_{j_m} or i_{j_n} and the integers $j_q < j_m < j_n$ represent the absolute positions, in S , of the symbols represented by i_{j_q}, i_{j_m} and i_{j_n} . If i_{j_q} represents the symbol $\aleph \in I - [A \cup P]$, i_{j_m} represents the symbol $\Im \in A \cup P$, i_{j_n} represents the symbol $\aleph \in A \cup P$ and if neither $\Im$ nor $\aleph$ is present in the S subsequence comprised between i_{j_q} and i_{j_m} , extremes excluded, and if neither $\Im$ nor $\aleph$ is present in the S subsequence comprised between i_{j_m} and i_{j_n} , extremes excluded, we call *example*, or *example of $\aleph$ relative to S* , the quintuple $E \equiv [i_{j_q}, i_{j_m}, i_{j_n}, \Delta m_q, \Delta n_m]$, where $\Delta m_q, \Delta n_m$, represents the relative position of i_{j_q} with respect to i_{j_m} , of i_{j_n} with respect to i_{j_m} , ie, it represents the number of symbols comprised, in the sequence S , between i_{j_q} and i_{j_m} , i_{j_m} and i_{j_n} , in the order. For short, we call Δmn the *distance* between m and n , in the order. We call *example father* the symbol represented by i_{j_q} ; *example first son* the symbol represented by i_{j_m} ; and *example second son* the symbol represented by i_{j_n} .

Let it be $i_{j_q} \equiv \aleph, i_{j_m} \equiv \Im, i_{j_n} \equiv \aleph, \Delta m_q = \Delta m'q', \Delta n_m = \Delta n'm'$, with n smaller than q' , let us consider the example $E' \equiv [i_{j_q'}, i_{j_m'}, i_{j_n'}, \Delta m'q', \Delta n'm']$. We call $\Delta E'E$ the relative position of the two examples E and E' , ie, the distance between i_{j_n} and $i_{j_q'}$. We call *consecutive* the examples E and E' , if in the sequence S there is no example $E'' \equiv [i_{j_q''}, i_{j_m''}, i_{j_n''}, \Delta m''q'', \Delta n''m'']$ with $i_{j_q''} \equiv \aleph, i_{j_m''} \equiv \Im, i_{j_n''} \equiv \aleph, \Delta m_q = \Delta m''q'', \Delta n_m = \Delta n''m'', i_{j_n}$ smaller than $i_{j_q''}$ and $i_{j_n''}$ smaller than $i_{j_q'}$, ie, if the sequence comprised between E and E' contains no example of $\aleph$ relative to S . Let us consider the subsequence S' belonging to the sequence S and containing S' symbols, with $\Delta m_q + \Delta n_m < S'$. We call *repetition cadence of the example E relative to the sequence S'* , or simply the *E repetition cadence*, for short Γ , the actual number of the E instances contained in S' . Let $\Theta = S'/[\Delta m_q + \Delta n_m]$ be the number of times, as opposed to the actual number of times, the example E could be contained in the subsequence S' and let t, u, v, w, θ be positive integers, with $\theta < \Theta$. By definition:

If there exists an S' symbol subsequence S' of the sequence S , such that S' first component i_{j_q} represents the symbol $\aleph$, if $\Theta^{-1} < \Delta E'E < \theta^{-1}$, ie, if the set of the distances between any two consecutive instances of the example $E \equiv [\aleph, \Im, \aleph, \Delta m_q, \Delta n_m]$ in S' has the upper bound θ^{-1} and the lower bound Θ^{-1} , ie, if the example E has a repetition cadence Γ , relative to S' , such that θ

$< \Gamma \leq \Theta$ and if $t < \Delta m q < u$, if $v < \Delta n m < w$, then, from the $[i_{jq} + S']$ -th symbol onward $\aleph \in A$ relative to the sequence S .

Obviously, for any given alphabet I , there are sequences S such that their relative sets A are empty. Eg, A is empty for the sequences obtained by the endless repetition of the very same symbol, for the sequences containing only either P or $I - P$ elements and for the sequences containing only examples with $\Gamma < \Theta$.

In the next chapters we propose the CH transform: by definition, the CH transform projects the symbols belonging to the subset P into a one-to-one correspondence with the P symbols belonging to a new, alphabet O . By contrast, no initial correspondence is given for the symbols belonging to the subset $I - P$, ie, initially CH is not defined on them. However, because the sequence S symbols are the CH independent variable values, as soon as some of them becomes the A elements, then, CH projects each instance of the A element $\aleph$ into the concatenation of the projections of the symbol $\aleph$ second and third sons, ie, into one well defined sequence of the O symbols. For later apparent reasons, we call *automatically adaptable* the said system. Actually, for an alphabet I containing tens of symbols the system we are proposing is quite elementary. By contrast, for our system the I cardinality is irrelevant. That is, CH complexity depends on the number of different projections, ie, on the number of different correspondences between the I symbols and the sequences of O symbols the system has to adapt to execute.

2. THE CH TRANSFORM.

2. 1. The Caianiello Transform.

In [1], we have presented a *CH transform*, namely the functional product of the Caianiello transform times the Hebb transforms, in the order. To start with, we define iteratively the *Caianiello*, or *C*, transform as:

$$f_{h, t+1} = 1[\sum_k C_{h, k} f_{k, t} - T],$$

where $0 < h \leq N$, $0 < k \leq N$, $0 < t$, $1[x]$ is equal to zero for $x < 0$ and to one otherwise. The function $f_{h, t}$ sometime is called the *state* of the binary vector F_t h -th component, accordingly the binary vector F_t is the *state vector*, the matrix element $C_{h, k}$ is called the *coupling coefficient* and the constant T the *threshold*. For simplicity sake, we say the binary vector F_t component h -th is *active at the time*, or *iteration*, t if $f_{h, t}$, ie, if the h -th component state is equal to one. Otherwise, we say it is *inactive*. As proved in [1, 2, 3, 4], the matrix $\|C_{h, k, t}\|$ elements can be chosen so that any $f_{h, t+1}$ computes the negation of the conjunction, ie, the NAND or the negation of the disjunction, ie, the NOR of its arguments $f_{k, t}$. For any given matrix $\|C_{h, k, t}\|$ and binary

vector F_t , the C transform projects them into the binary vector F_{t+1} . In other words, as it maps the binary vector set into itself, the C transform can be applied to its own result to generate a *binary vector succession*.

2. 2. The Hebb Transform.

Now let us suppose the set C of the matrix $\|C_{h, k, t}\|$ elements to be composed by two mutually exclusive subsets. One contains the *time, or iteration, independent elements* $C_{h, k}$, for short the *constant elements*, and the other contains the *time, or iteration, dependent elements* $0 < C_{h, k'}, t < T$, for short the *variable or Hebbian elements*. For simplicity sake, we call k' the index of the generic time dependent element. We call *Hebbian-in* the binary vector h -th component and *Hebbian-out* the binary vector k' -th component, whenever the coupling coefficient $C_{h, k'}, t$ is Hebbian. Obviously, a binary vector component can be either non-Hebbian or Hebbian-in or Hebbian-out or both Hebbian-in and Hebbian-out. In any case the Hebbian is a transient initial property: it cannot be acquired, it can be only lost as t increases.

We define the *Hebb, or H, transform* as follows:

i) given the two binary vectors F_t and F_{t+1} ; ii) given the *Hebbian parameters*: "b", "d", "e" and "g" greater than one; iii) if $\sum_{k'} C_{h, k'}, t f_{k'}, t$ is smaller than the threshold T , where $\sum_{k'}$ represents the sum bounded to all the k' of the time dependent matrix elements $C_{h, k'}, t$, then, the *Hebbian transform* is defined by the relations:

$$\begin{aligned}
 C_{h, k', t+1} = & \\
 & C_{h, k', t} - b && \text{if } f_{h, t} = f_{k', t-1} = 0 \text{ and } 0 \leq C_{h, k', t} - b, \\
 & - [C_{h, k', t} - b] && \text{if } f_{h, t} = f_{k', t-1} = 0 \text{ and } C_{h, k', t} - b < 0, \\
 & C_{h, k', t} - d && \text{if } f_{h, t} = 1 - f_{k', t-1} = 0 \text{ and } 0 \leq C_{h, k', t} - d \text{ and} \\
 \sum_{k'} C_{h, k', t} * f_{k', t-1} < T, && \\
 & - [C_{h, k', t} - d] && \text{if } f_{h, t} = 1 - f_{k', t-1} = 0 \text{ and } C_{h, k', t} - d < 0 \text{ and} \\
 \sum_{k'} C_{h, k', t} * f_{k', t-1} < T, && \\
 & C_{h, k', t} - e && \text{if } f_{h, t} = 1 - f_{k', t-1} = 1 \text{ and } 0 \leq C_{h, k', t} - e, \\
 & - [C_{h, k', t} - e] && \text{if } f_{h, t} = 1 - f_{k', t-1} = 1 \text{ and } C_{h, k', t} - e < 0, \\
 & C_{h, k', t} + g && \text{if } f_{h, t} = f_{k', t-1} = 1 \text{ and } \sum_{k'} C_{h, k', t} * f_{k', t-1} < T, \\
 C_{h, k', t+1} = C_{h, k', t} = C_{h, k'} \text{ and } C_{h', k', t+1} = C_{h', k'} = 0, && \text{for the given } k' \text{ and } \forall h' \\
 \neq h, && \\
 && \text{if } f_{h, t} = f_{k', t-1} = 1 \text{ and } f_{h', t} = 1 - f_{k', t-1} = 0 \text{ and} \\
 \sum_{k'} C_{h, k', t} * f_{k', t-1} \geq T. &&
 \end{aligned}$$

To sum it up, if either the Hebbian-in component h , or the Hebbian-out component k' , or both, are inactive, the variable coupling coefficient $C_{h, k', t}$ is decreased. In any case, a variable matrix element is decreased as long as it stays non negative. That is, it does never assume negative values. If both the Hebbian-in component h is active at the time t and the Hebbian-out component k' has been active at the time $t - 1$, for $0 < t$, and if the sum of the time dependent coupling coefficients having the indices h and k' is smaller than the threshold T , then, the time dependent coupling coefficient $C_{h, k', t}$ is increased. Otherwise, namely if both the component h is active at the time t and the component k'' has been active at the time $t - 1$ and if the latter sum is no smaller than T , then, from the next iteration on, the coupling coefficient $C_{h, k'', t}$ becomes the time independent constants $C_{h, k''}$ and all the remaining Hebbian coupling coefficients having the first index different from h and the second index equal to k'' are set to zero, ie, become time independent constants.

In the last clause we have set to zero the time dependent coupling coefficients having as first index the index of the Hebbian-in component inactive at the time t and as second index the index of the Hebbian-out component active at the time $t - 1$. By contrast, if the said coupling coefficients become constants at their greater than zero unknown values we could have, depending on the said unknown values, unpredictable results. E. g., given the Hebbian-out components k^* and k^{**} and the Hebbian-in components h' , h'' and h''' , if for a sufficient number of iterations it is:

$$\begin{aligned} f[h', t] &= f[k^*, t - 1] = 1 \\ f[h'', t] &= f[k^{**}, t - 1] = 1, \end{aligned}$$

thanks to the given Hebbian rule we obtain the greater than the threshold C_{h', k^*} and $C_{h'', k^{**}}$ constant values, for all t greater than a t^* . However, if we do not set to zero the remaining coupling coefficients, we could obtain also the constant values $C_{h''', k^*} + C_{h'', k^{**}} > T$, , for all t greater than a t^* , even if not once we had for t smaller than t^* :

$$f[h''', t + 1] = f[k^*, t] = f[k^{**}, t] = 1.$$

That is, we could end with the component h''' computing the logical conjunctions of the components k^* and k^{**} activity, even if for t smaller than t^* not once it has ever computed such a function. Because such result depends on the C_{h'', k^*} and $C_{h'', k^{**}}$ unknown values, to have H transforms with predictable behaviors, we have chosen the given Hebbian rule version.

For any given matrix $\|C_{h, k, t}\|$ and binary vectors F_t and F_{t+1} , the Hebbian transform projects them into the matrix $\|C_{h, k, t+1}\|$. In other words, as it maps the matrix set into itself, the Hebb transform can be applied to its own result to generate a *matrix succession*.

2. 3. Some Of The CH Transform Properties.

For any given matrix $\|C_{h, k, t}\|$ and binary vector F_t , the CH transform projects, in the order, the binary vector F_t into the binary vector F_{t+1} and the matrix $\|C_{h, k, t}\|$ into the matrix $\|C_{h, k, t+1}\|$. In other words, as it maps the binary vector and the matrix sets into themselves, the CH transform can be applied to its own result to generate both a binary vector succession and a matrix succession.

We shall say: the vector and matrix successions have *definitively acquired the property relative to k at the iteration, or time, t^** if, first, the vector k-th component which is Hebbian for t smaller than t^* is non-Hebbian for t no smaller than t^* , second, the matrix k-th column elements which are time dependent for t smaller than t^* are constant for t no smaller than t^* .

In [1, 3, 4] we have proposed some CH transform instantiations, namely we have proposed the Hebbian parameters together with some initial matrices and a code from the alphabet I into the states of a subset of the binary vector components: let us call I the binary subvector having as components the elements of the latter subset. If I has seven, eight, components, we take as code the standard seven, eight, bits ASCII code. Eventually, thanks to the given code, the sequence S symbols are in a one-to-one correspondence with the succession of the I subvectors, one per iteration.

Accordingly, we say the symbol $\aleph$, belonging to the alphabet I, *identifies the binary subvector $I_{\aleph}$* , if $\aleph$ code word is represented by the $I_{\aleph}$ component values. In general, we say the binary vector components with the indices $k_1, k_2, \dots, k_j$, with j a given positive integer, *identify the component with the index h*, if $C_{h, k_1} > T/j, C_{h, k_2} > T/j, \dots, C_{h, k_j} > T/j$, ie, if $f_{h, t+1}$ computes the logical AND of $f_{k_1, t}, f_{k_2, t}, \dots, f_{k_j, t}$. We shall say the binary vector *component k identifies the component h with a delay d*, if there are d - 2 sorted components such that the component k identifies the first of them and each of them identifies the next one and the last of them identifies h. A component can identify two, or more, different components, each with a different delay.

The P symbols $i_1, i_2, \dots, i_P$ are coded into the states of the binary vector components belonging to a proper subset of the I subset. Let us call $k_1, k_2, \dots, k_p, \dots, k_P$ the indices of the binary vector components whose values are the code words for the P elements. In every initial matrix proposed in [1, 2, 3, 4], there are the P elements $C_{h_p, k_p} > T$, for $0 < p \leq P$, that is, the complement of I on the binary vector contains the P *primitive* components, with the indices $h_1, h_2, \dots, h_p, \dots, h_P$, such that:

$$f_{hp, t+1} = f_{kp, t}$$

That is, because of the matrices we have proposed, each primitive component becomes active one iteration after its corresponding component belonging to the I subvector has been active, the I remaining components being inactive. In other words, each of the I symbols belonging to the P subset identifies one of the binary vector primitive components. We call *output*, or *O, subvector* the vector composed by the P primitive components and *output sequence* a sequence of finitely many O subvectors.

3. AN AUTONOMOUSLY ADAPTABLE SYSTEM.

Let us consider one of the proposed CH transform instantiations, see [1, 3, 4], and a sequence S containing a finite number of finite subsequences, in turn, containing at least one example each with a repetition cadence verifying the given conditions. That is, the sequence S contains both the subsequence S' which, in turn, contains at least the example E' with a repetition cadence Γ' and the subsequence S'' which, in turn, contains at least the example E'' with a repetition cadence Γ'' and ... and the subsequence S''' which, in turn, contains at least the example E''' with a repetition cadence Γ''' . In the proposed CH transforms, we have chosen the Hebbian parameters so that each example E''' with the repetition cadence Γ''' is sufficient to increase the value of at least some of the matrix iteration dependent elements until they become constants. Accordingly, every time an I symbol becomes an element of the set A, relative to the sequence S, the I binary subvector representing its code word identifies one component belonging to the I complement on the binary vector: as the said element of the set A is the first symbol of the example father, let us call the latter component the *father component*. In turn, because of the initial values we have chosen for the matrix elements, in the proposed CH transform, the father component identifies the two components identified by the example first and second son, with the respective delays equal to their distances in the sequence S. As the last conclusion can be iterated for each of the latter two components and for the components identified by the example sons and for the components identified by the example sons of their example sons and so on, if at the iteration t^* the I binary subvector components activity is the code word of a symbol belonging to the set A, for t greater than t^* , the very same symbol identifies a well defined output sequence, see the definition of the example relative to S.

We say: **the system autonomously has adapted itself to the symbol $\aleph$ at the time t^* relative to a sequence S, if it is $\aleph \notin A$ for $t < t^*$ and from t^* onward $\aleph \in A$ relative to S.** In other words, as a system has autonomously adapted itself to a symbol $\aleph$ at the time t^* relative to a

sequence S , whereas to each instance of the system $I - P$ binary subvector components activity representing the symbol $\aleph \notin A$ at any time smaller than t^* it does not correspond one output sequence, from t^* onward to each instance of the very same $I - P$ binary subvector components activity representing $\aleph \in A$ relative to S it does correspond one well defined output sequence. Actually, the output sequence corresponding to the symbol $\aleph \in A$ is the one obtained as the concatenation of the output sequences corresponding to $\aleph$ first and second son. Because the output sequences are O subvector sequences, the choice of a code from the primitive component states into an output alphabet is fundamental for the system universality.

The lengths of the output sequences so identified have an upper limit which depends on the CH vector, and hence matrix, dimension. However, whereas it is true that the output sequence lengths have an upper limit, there are no limits on the number of the output sequences which can be chained one after the other, provided we can project the output subvector component values into the input subvector component values. For some illustrative examples on how to project the former into the latter subvector component values and on how to chain different output sequences, see [1, 2].

Eventually, if in the sequence S there are triplets of symbols which do not satisfy the example conditions, eg the example father belongs to set $A \cup P$ or the same symbol is both the father component and one of the two sons, then a loop could be introduced into the system. Or to verify if the said conditions are always verified, eg whenever the set A is empty both sons have to be P elements, we have found initial values for the proposed matrix elements, see [4]. In other words, among the binary vector components, belonging to the $I \cup O$ subvector complement, we have inserted some computing suitable algorithms, ie, computing suitable combinations of the NAND and the NOR of their arguments to prevent the system from adapting to the triplets which do not satisfy the example conditions.

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In ricordo di Giovanna Misuraca (1937-1999) e dei suoi studi sulla melanogenesi.

The pigment of Sepia: an analytical approach

Nota di Marco Olivieri e del socio Rodolfo A. Nicolaus

(Seduta del 6 novembre 1999)

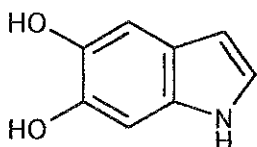
Key words: eumelanin, dopachrome, cuttlefish, melanosome.

Sepiomelanin is not the most elusive and enigmatic pigment found in Nature but just a particle waiting for a rational study.

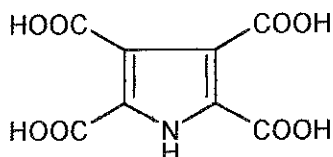
The ink ejected by octopus and squid to deceive attacking fish is a black suspension of melanin particles (melanosomes). The ink of the cuttlefish, *sepia*, may be found at different degree of density. Sepiomelanin Na⁺ form occurs in the ink sac of *Sepia officinalis* as Ca and Mg salt. The free acid is obtained by action of dilute HCl: the sepiomelanic acid. Sepiomelanin is a black insoluble, infusible, hygroscopic powder; in fresh preparations, on the contrary, loss of weight (loss of CO₂?) is observed. The crude ink is solubilized by sonication at pH 10 and 15 min. at 80 W [12]. The pigment is a mixture of particles in varying stages of development as shown by ultracentrifugal density-gradient methods [1]. Working with fresh ink a *post mortem* melanogenesis in still active organelles occurs, resulting possibly in an artificial melanogenesis of dopachrome. Heating of sepiomelanin produce loss of CO₂ and of H₂O. Melanin is sensitive to oxygen, oxidants, pressure, light; pulsed laser irradiation produce an explosive fragmentation of melanosome similar to that occurring in the skin [2].

The pigment has the property to bind many organic products, drugs, ions, gas, recalling the behaviour of charcoal. Some properties like the EPR spectrum are in agreement with a radical-polarone polymer structure. Sepiomelanin-film in doped or undoped form acts as an amorphous semiconductor. As a Little's

structure it may conduct electricity without resistance like certain metals [3], [4], [5]. Chemical and physical studies show that sepiomelanin, in contrast with general view, is derived from 5,6-dihydroxyindole (DHI).



DHI

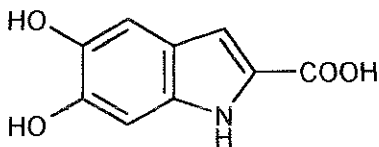


Pyrrole-tetracarboxylic acid

Polymerization through all free positions of DHI produces a polycondensed structure and further assembling of intermediates build up the particle [6]. The presence of polycondensed structure is in agreement with X-rays diffraction spectra and, chemically, with the isolation of pyrrole-tetracarboxylic acid among the oxidative degradation products of sepiomelanin.

More recent studies have shown that the ejected ink of *Sepia officinalis* contains an enzyme, which promotes the decarboxylative rearrangement of dopachrome to DHI [7], [8].

The study of sepiomelanin isolated and purified with incorrect methods produced results contrasting the assumption that the black particle is constructed with DHI units. For example, some results have shown that sepiomelanin is a copolymer of DHI and DHICA (5,6-dihydroxyindole-2-carboxylic acid) [9].



DHICA

More recently, it has been suggested that sepiomelanin is a mixture of oligomeric structures incorporating over 75% of DHICA-derived units and only 20% of DHI-derived units, occurring for the most part as pyrrolecarboxylic acid degraded form [10]. It should be noted that such a structure cannot be black, would not show the typical EPR spectrum and would be easily soluble in dilute

alkaline solutions.

These conflicting and surprising results are caused probably by the incorrect methods of isolation and purification adopted. In order to avoid these inconvenients we present a protocol for isolation and purification of sepiomelanin based chiefly on Chedekel method [11].

Considering that all sepiomelanin samples examined so far are artifacts, all literature data should be reexamined. Some papers, particularly, [11], [9], [12], [13], [10], [14], [15], [16], [17], [18], require a large critical discussion. Sepiomelanin protocol of isolation and purification starting from fresh ink is given in the following table 1. Variable amounts of H_2O_2 containing melanosomes are present in fresh ink. How and when to eliminate the H_2O_2 has not been considered so far. Extraction in the presence of catalase of the naturally ejected ink may be recommended.

Reagents are analytical grade. Distilled water was used. All operations are conducted at room temperature working in the dark. Solids are separated from liquids by centrifuging. Samples for analysis are dried on P_2O_5 in vacuum at room temperature. Working at high centrifuge speed a large amount of immature melanosomes can be obtained. The mature/immature melanosome ratio was found to change in stored and in fresh ejected ink. Centrifuge speed (2000 - 4000 rpm) is adjusted with the ink fluidity.

Melanosomal fractions of different maturation grade, shape and size may be obtained by following methods:

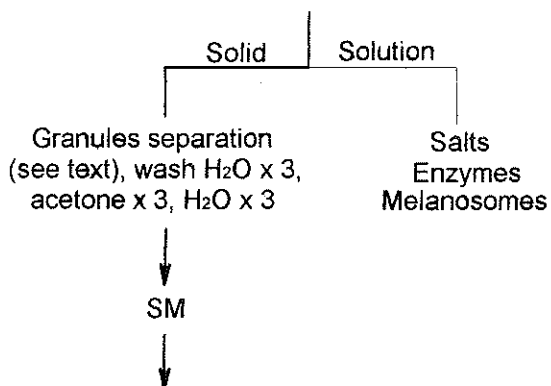
- a) by varying the centrifugation speed
- b) the "sieve" procedure of Chedekel [11]
- c) the "sugar" procedure of Hearing [19]
- d) the "surfactants" procedure of Ito-Prota [10].

MALDI (matrix assisted laser desorption ionization) spectra may be a good distinctive criterium between the different melanosomal fractions. In the MALDI spectrum of immature melanosome, precursors and oligomers are present, whereas they are absent in mature melanosome.

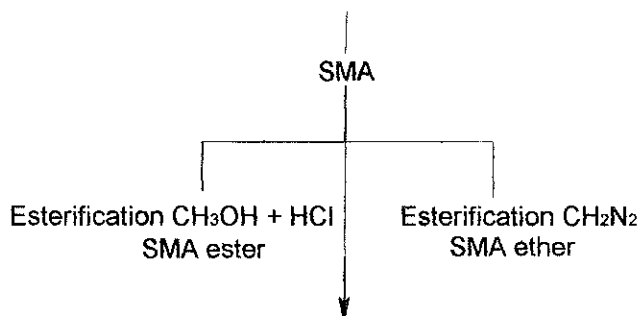
With some modification, the above mentioned procedure (table 1) could be applied to the study of others melanins.

Table 1

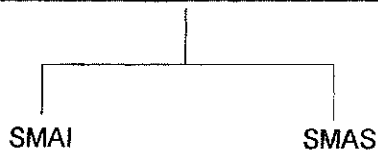
An ink sac was cut open, emptied and centrifuged: wash with H₂O x 3



Wash HCl 1N x 3, keep 1 night HCl 6N, wash HCl 1N x 3, H₂O x 3, acetone x 3, H₂O x 3. Dry on P₂O₅ in vacuum.



Treatment with H₂O₂ in Na₂CO₃ 2N. Filter if necessary. Acidify with HCl 2N. Filter and centrifuge.



Legend for table 1

SM = Ca, Mg salt of SMA, homogeneous (size, density, mass spectrometry) fraction of mature melanosomes (granules) from *Sepia* ink. Samples for analysis are dried at room temperature over P_2O_5 . Analysis for C, H, N, Ca, Mg, Na, Fe, Cu, Zn.

Physical determination:

- IR
- EPR
- X-rays
- MALDI
- Conductibility
- Electronic microscopy

SMA = Sepiomelanic acid obtained from SM. Samples for analysis are dried at room temperature over P_2O_5 . Analysis for C, H, N, Ca, Mg, Na, Fe, Cu, Zn.

Physical determination:

- IR
- EPR
- X-rays
- MALDI
- Conductibility

SMA ester = methyl ester of SMA obtained by esterification with $CH_3OH + HCl$ gaseous [9].

SMA ether = methyl ester (ether) of SMA obtained by CH_2N_2 treatment [9].

SMAI = Insoluble sepiomelanic acid obtained after oxidation with H_2O_2 in alkali of fine powdered SMA and acidification of alkaline solution [20]. Simple

and complexe pyrrolepolycarboxylic acids may be present in this precipitate. SMAI was obtained in 70% yield w/w. This acid is indicated for chemical and physical studies thanks to its solubility in alkali.

SMAS = Sepiomelanic acid solution obtained from SMA after oxidation with H_2O_2 in Na_2CO_3 2N, acidification and separation from SMAI [20]. Simple acids, carbonilic compounds, simple and complexe pyrrolepolycarboxylic acids are present in this solution.

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In memoria di Marsden Scott Blois Jr. (1919-1988) che guardò per primo alla melanina come una matrice molecolare dei sistemi prebiologici.

Riflessioni sulla chimica della materia nera interstellare

Nota di Bruno J. R. Nicolaus, Rodolfo A. Nicolaus e Marco Olivieri

(Adunanza del 6 Novembre 1999)

Key words: melanins, fullerene, cosmoids, cosmochemistry, X-rays, LASER, black particles, interstellar matter, Bok globules.

Il materiale nero è universalmente diffuso soprattutto allo stato amorfo. Tutto il materiale nero finora conosciuto, dalla litosfera e biosfera al cosmo, è in genere caratterizzato da proprietà chimiche e fisiche comuni. Proprietà comuni sono: l'EPR (Electronic Paramagnetic Resonance); le proprietà elettriche; la modifica delle proprietà di superficie sotto l'azione del campo elettrico; lo spettro di diffrazione dei raggi X [1]; la sensibilità alle radiazioni che produce fotoionizzazione e fotolisi [2]; l'esplosione e la frammentazione per effetto del LASER, della pirolisi, del bombardamento atomico veloce [3, 4, 5]. In effetti le proprietà fisiche (e in parte chimiche) delle particelle nere non dipendono da struttura e natura del melanogeno.

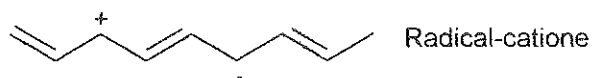
In tutti i pigmenti è possibile riconoscere un esteso sistema policoniugato radical-polaronico detto spina di Little [6]. Gli atomi fanno

parte di esagoni e di pentagoni che sono assemblati in sandwich di strati distanziati $3.4 \text{ \AA}$ tipo grafite (forma aperta) o in fullereni giganti (forma chiusa). Proprietà chimiche rilevanti del materiale nero sono la sensibilità all' H_2O_2 e agli alogeni, la perdita irreversibile di CO_2 e H_2O per riscaldamento, la capacità di legare prodotti organici, droghe, ioni e gas. Sotto molti aspetti le melanine ricordano nel loro comportamento chimico-fisico quello del carbone attivo.

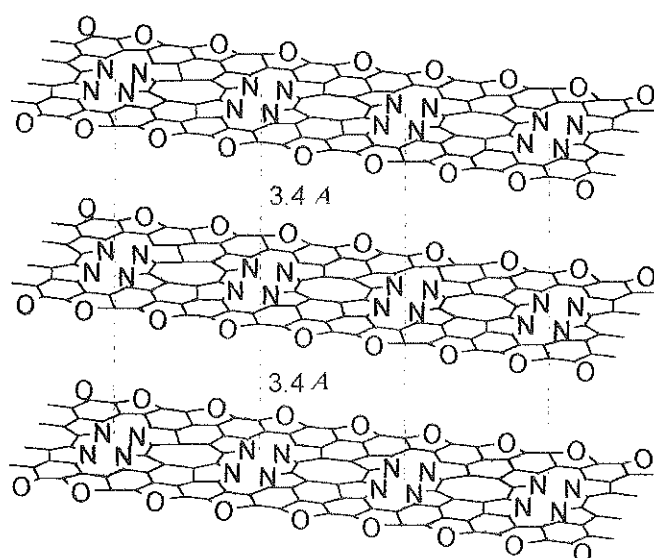
La radioastronomia ha mostrato la presenza di molecole organiche tra le nubi di polvere nera presenti nella Via Lattea: molecole a volte molto semplici come l'acetonitrile o sistemi acetilenici come i poliini. Molecole acetileniche come HC_5N , HC_7N , HC_9N sono state individuate per mezzo dell'analisi spettroscopica. E' stato anche osservato che le stelle giganti rosse producono enormi quantità di polveri carboniose nello spazio suggerendo un collegamento tra le suddette strutture acetileniche e la formazione della fuliggine nera [7, 8]. In altre parole lo spazio interstellare può risultare nero non solo a causa della mancanza di fonti luminose o per l'esistenza di forti campi gravitazionali che impediscono alla luce di uscirne [9] ma anche per la presenza di materiale nero allo stato solido. Quest'ultimo sarebbe, sotto l'azione delle radiazioni, in continua trasformazione.

Com'è noto la grafite, presente negli spazi interstellari, sotto l'azione del LASER in un ambiente che simula quello di alcune zone cosmiche, si frantuma e produce una serie di fullereni [7b] tra cui il famoso C_{60} . Analogamente le particelle nere e le melanine vengono frantumate dal LASER. Il LASER, usato in spettrometria di massa (MALDI-TOF), non

fornisce picchi molecolari relativi alle melanine ma dei sottoprodotti che potrebbero essere di interesse per la cosmochimica [5].



Radical-polarone come struttura comune a tutte le melanine e causa della conduttività o superconduttività.



Siti porfirino-simili e clatrati della struttura grafica a sandwich presente nelle melanine

Nella figura 1 si può osservare un'immagine fotografica di nubi interstellari oscure disperse tra la moltitudine di nebulose che affollano la

costellazione del Centauro. Tali nubi note come “globuli di Bok” sono generalmente associate ai processi di formazione protostellare [10, 11].

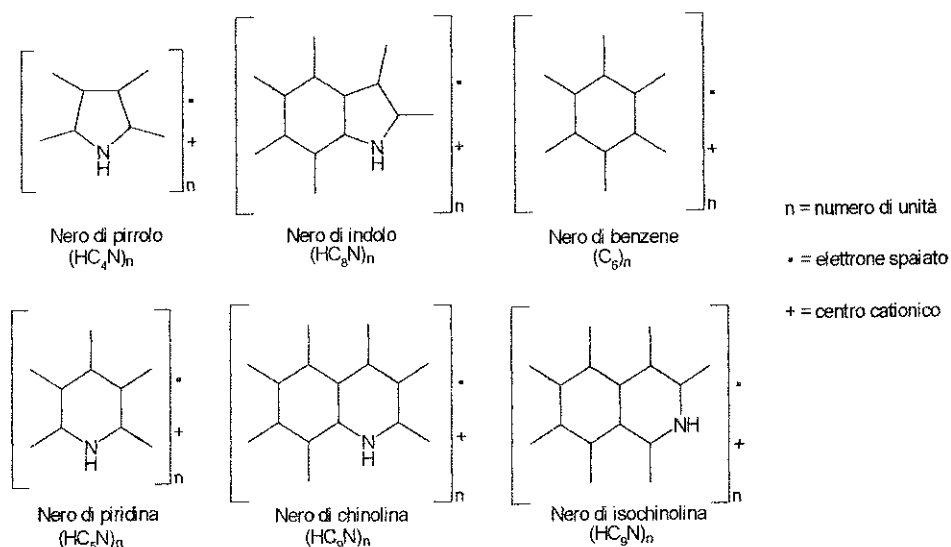
Figura 1



Nelle nubi scure spaziali potrebbero essere presenti particelle nere formate da strutture benzeniche o eterocicliche policondensate come nel caso delle melanine terrestri. L'emissione di microonde da parte di tali nubi ha inoltre mostrato una relativa abbondanza di composti. Infatti, è stata riscontrata la presenza di oltre 100 molecole differenti da H_2 e CO (tra le più abbondanti) fino a molecole tipo CH_3NH_2 , CH_3OH , CH_3CH_2OH , HC_6CN , HC_9N , $HNCO$, CH_3CN , CH_2CHCN , CH_2CO , H_2CO , $HCNS$, CS , COS , CH_3SH , CH_2S , acetilene, etilene, metano, silano, composti acetilenici, sistemi aromatici policondensati [7, 12, 13, 16], grafite [9 pag 630]. Le diverse molecole potrebbero, in parte, essere

prodotti dell'esplosione di sistemi policondensati tipo quelli rappresentati nella figura 2 in modo simile a quanto avviene in laboratorio per pirolisi o bombardamento atomico [4]. Le molecole solforate ed ossigenate, analogamente, potrebbero provenire da sistemi tipo tiofene o furano.

Figura 2

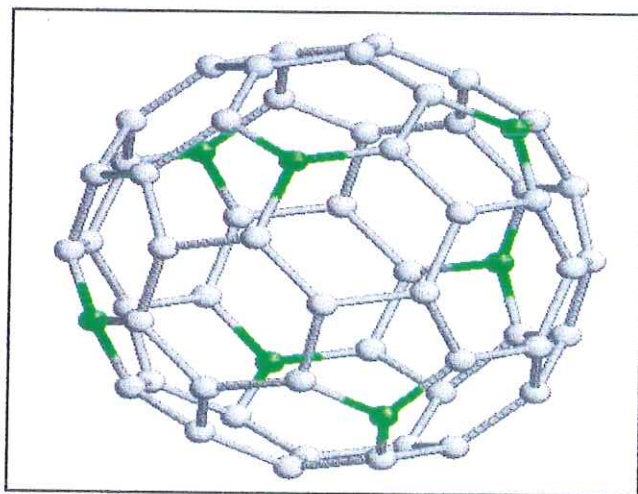


Miscele di questi materiali si trovano in vari tipi di catrami, sono formati da anelli esagonali e pentagonali e sono dei conduttori elettrici poco conosciuti (ad eccezione del nero di pirrolo) [14] a causa delle difficoltà di isolamento e purificazione. La spettroscopia e la spettrofotometria di questi materiali possono essere di ausilio nello studio della cosmochimica e del materiale interstellare nero.

E' generalmente riconosciuto che l'evoluzione biologica sia stata preceduta da quella chimica a livello molecolare ed è stata suggerita, in relazione a ciò, una probabile sintesi di sostanze porfirino-simili durante

l'era prebiotica [15]. La presenza di siti porfirino-simili [1b] nelle melanine lascia supporre un ruolo di tipo catalitico, in simbiosi con i metalli, svolto dalle particelle nere presenti sulla terra, le quali, rispetto ai minerali, sarebbero i candidati ideali nel rappresentare un prototipo di struttura che ha giocato un ruolo strutturistico nell'era preenzimatica (siti stereospecifici, clatrati, capacità di legare metalli, fotoprotezione, ecc.). Inoltre, un ruolo delle particelle nere nella formazione dell'atmosfera terrestre primordiale può essere stato giocato grazie alla loro particolare capacità di intrappolare e cedere gas (strutture a gabbia di tipo fullerene gigante?). La sostituzione di alcuni atomi di carbonio con atomi di azoto in un modello molecolare computerizzato di C_{60} conduce alla formazione di un "fullerene azotato" $C_{53}N_7$ in cui sono presenti unità chinoliniche. Nella figura 3 è possibile notare che l'ottimizzazione geometrica effettuata al calcolatore non ha prodotto deformazioni nella struttura iniziale di tipo soccer-ball. Per i fullereni giganti si prevedono spettri di diffrazione dei raggi X simili a quelli previsti per strutture grafitiche.

Figura 3



La particella nera in generale costituisce un fattore interessante nell'evoluzione della materia interstellare e biologica. Le particelle nere sono conduttori. Questa proprietà può consentire il trasferimento di cariche elettriche all'interno delle nubi molecolari regolando la chimica e gli equilibri ione-radical-molecola. Fatti d'altronde osservabili attraverso l'assorbimento, la polarizzazione della luce stellare e l'analisi spettrale dell'emissione IR della polvere [9, 12]. D'altra parte il ruolo delle particelle nere potrebbe essere inquadrato nell'ambito dell'evoluzione prebiotica sia come strutture assemblatrici di atomi e molecole sia come generatrici di altre specie molecolari in seguito ad annichilimento. La presenza di onde d'urto prodotte dalle supernovae potrebbe, infatti, così come avviene con i grani interstellari, quando queste superano i 25km/sec, fare esplodere le particelle nere presenti nello spazio. Ioni ad alta velocità possono produrre frammentazione e formare miscele di prodotti semplici. E' notevole il fatto che alcune molecole organiche semplici che si trovano nelle nubi nere [12] si ritrovano anche tra i prodotti della pirolisi e del bombardamento veloce atomico della sepiomelanina, melanina dei capelli, tirosina-melanina, serotonina-melanina, triptamina-melanina [4]. Sono stati identificati il CH_4 , CO_2 , H_2O , HCN , CH_3Cl , CH_3OH , C_4H_8 , C_2H_6 , COS , CH_3CHO , CH_3SH , C_4H_{10} , H_2S , CH_3CN , C_3H_6 , metilcicloesene, piridina, toluolo, metilpirrolo, etenilbenzene, fenolo, acido benzoico, fenilestere, benzaldossima, metilindolo, 2,3-dimetilcicloesanol, 1,2-benzoisotiazolo, benzaldeide. Con le missioni spaziali Stardust, Space Technology 4/Champlion, Rosetta, che invieranno dati ed analisi della polvere nera nel 2005, 2006, 2013 [16], avremo informazioni più dettagliate. I dati da

noi presentati insieme a quelli che si potranno raccogliere dallo studio spettroscopico dei materiali neri terrestri (catrami, melanine, neri di sintesi) potrebbero essere utilizzati nell'analisi dei campioni raccolti nelle varie missioni.

I chimici vedono la materia nera interstellare caratterizzata da strutture aromatiche ed eterocicliche policondensate alcune delle quali conosciute sulla terra. La materia nera sotto l'azione delle radiazioni esplode con formazione di frammenti più piccoli facilmente individuabili. Il materiale nero è in continua evoluzione e trasformazione con produzione di molecole organiche a basso peso molecolare che vengono trasformate in ghiaccio. Il nero, come conduttore elettrico, può regolare la chimica e gli equilibri ione-radical-molecola all'interno delle nubi interstellari. Il materiale nero esercita un'azione protettiva dalle radiazioni sul materiale organico. Le energie fondamentali (termica, radiazione ionizzante, radiazione ultravioletta) per la sintesi organica vengono fornite dalle stelle. Si prevede che i frammenti più piccoli provenienti dall'esplosione della materia nera daranno luogo a molecole organiche simili e/o uguali a quelle già conosciute sulla terra.

La cosmochimica della materia nera potrebbe far rinascere sulla terra l'interesse per lo studio delle melanine, sviato fino ad oggi da piste false. Superato lo scoglio della purificazione ed estrazione [17] si potrebbero sviluppare in modo interessante, come già in atto, la nanochimica, nanobiochimica e nanofisica delle melanine (particelle nere) [5, 18-26].

La melanina è un conduttore a configurazione variabile sotto l'azione di un campo elettrico od elettromagnetico. La melanina assembla gli elementi più semplici, può controllare la forma e la funzione di aderenza

cellulare [18,19]. La melanina stellare produttrice di molecole organiche, la melanina terrestre assemblatrice di molecole e macromolecole organiche.

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Versione inglese :

Speculation on the chemistry of interstellar black matter

Black matter is found universally, especially in the amorphous state. All the black matter known to date, from the lithosphere and biosphere to the cosmos, generally has the same chemical and physical properties. These include electron paramagnetic resonance (EPR); electrical properties; changes in surface properties under the effects of an electric field; the X-ray diffraction spectrum [1]; sensitivity to radiation inducing photo-ionization and photolysis [2]; explosion and fragmentation under the effects of lasers, pyrolysis, or fast atom bombardment [3-5]. The physical properties of black particles, and some of the chemical ones too, do not depend on the structure and nature of the melanogen. An extensive polyconjugated radical-polaronic system can be found in all pigments, known also as Little's spine [6]. The atoms are arranged in hexagons or pentagons, which are assembled in sandwiches in layers 3.4 Å apart, like graphite in the open form, or huge giant fullerenes in the closed form. Black material has certain striking chemical properties: it is sensitive to H₂O₂ and halogens, it irreversibly loses CO₂ and O₂ on heating; it can bind organic products, drugs, ions and gas. From many viewpoints the melanins resemble activated charcoal in their chemical-physical behaviour.

Radioastronomy has shown there are organic molecules in the black dust clouds in the Milky Way. Polyynes like HC₅N, HC₇N, and HC₉N have been detected by spectroscopic analysis. Giant red stars also emit

enormous amounts of carbon dust into the surrounding space, suggesting there may be some links between the acetylene structures and the soot formation [7, 8]. This implies that interstellar space may look black not just because of the lack of light, or because strong gravitational fields prevent light escaping [9], but also because of the presence of black matter in the solid state. This matter would be in continual transformation under the action of radiation.

Graphite(which is found also in interstellar space) breaks up under the action of LASER rays in a setting simulating certain parts of the cosmos, producing a series of fullerenes [7b], including the well-known C_{60} . LASER light also fragments black particles and melanins. The LASER in mass spectrometry (MALDI-TOF) does not give molecular peaks for melanin, but there are subproducts which might be interesting in astrochemistry [5].

Fig. 1 shows a photograph of dark interstellar clouds among the nebulae crowding the constellation of the Centaurus. These clouds, known as "Bok globules", are normally associated with protostellar formation [10, 11]

The dark clouds in space may be made of black particles with aromatic polycondensate structures like terrestrial black pigments .A relative abundance of low-molecular-weight organic compounds has been detected in the microwaves emitted by these clouds. More than 100 different molecules have been found, H_2 and CO being among the most abundant; molecules like CH_3NH_2 , CH_3OH , CH_3CH_2OH , HC_6CN , HC_9N , $HNCO$, CH_3CN , CH_2CHCN , CH_2CO , H_2CO , $HCNS$, CS , COS , CH_3SH , CH_2S , acetylene, ethylene, methane, silane, acetylene

compounds, polycondensed aromatic systems, hexamethylenetetramine (HMT), porphyrins, microdiamonds, amorphous carbon [7, 12, 13, 16] and graphite [9, p.630]. The different molecules might be partly produced by the explosion of polycondensate systems like pyrrole-black, indole-black, benzene-black, pyridine-black, quinoline-black, isoquinoline-black (Fig. 2) similar to the laboratory procedures for pyrolysis or atom bombardment [4]. Sulphated and oxygenated molecules might come, similarly, from thiophene or furane polycondensate systems.

Mixtures of these materials are found in tars; they are made up of hexagonal and pentagonal rings and - except for pyrrole black - are little-known electrical conductors [14] on account of difficulties in isolating and purifying them. Spectroscopic and spectrophotometric analysis of these material is a potentially useful way of investigating cosmochemistry and interstellar black matter.

It is generally acknowledged that biological evolution followed on the heels of molecular chemical evolution, and this has led to the proposal that there was probably some synthesis of porphyrin-like substances in prebiotic times [15]. The presence of porphyrin-like sites [1b] in the melanins suggests that the black particles found on the earth had some sort of catalytic role, in symbiosis with metals. The black particles may also have played a part in forming the primordial atmosphere on earth on account of their ability to trap and release gases (in a giant fullerene "cage-like" structure ?). Giant fullerenes would probably have X-rays diffraction spectra similar to those for graphite structure.

Black particles in general are an interesting feature in the evolution of interstellar and biological matter. They are conductors, which means they

can transfer electrical charges from inside molecular clouds, regulating their chemistry and the ion-radical-molecule equilibria. These properties can be deduced from the absorption, polarization of stellar light and spectra of the infrared light emitted by dust [9, 12]. The black particles might also have played a part in prebiotic evolution as atom and molecule assembly structures, or as generators of other molecular structures that have been annihilated. The shock waves produced by the supernovae might have the effect - like interstellar particles when they move faster than 25 km/second - of making the black particles in space explode. High-speed ions can cause fragmentation, and mixtures of simple products form. Some of the simple organic molecules found in black clouds [12] are also among the products of pyrolysis and fast atom bombardment of sepiomelanin, melanin from hair, tyrosine-melanin, serotonin-melanin, and tryptamine-melanin [4]. The following compounds have been identified: CH_4 , CO_2 , H_2O , HCN , CH_3Cl , CH_3OH , C_4H_8 , C_2H_6 , COS , CH_3CHO , CH_3SH , C_4H_{10} , H_2S , CH_3CN , C_3H_6 , phenol, methylcyclohexene, pyridine, toluene, methylpyrrole, ethenylbenzene, benzaldehyde, benzoic acid, phenylester, benzaldoxime, 2,3-dimethylcyclohexanol, 1,2-benzisothiazol, methylindole. The Stardust, Space Technology 4/Champion, and Rosetta space missions should transmit analytical data on black dust in the years 2005, 2006 and 2013 [16] so we should then have more details. The information presented here, together with whatever we find out from studying terrestrial black matter - tar, melanin, synthetic black - could be helpful in analysing the samples collected in the various missions.

We see interstellar black matter as having aromatic and polycondensed heterocyclic structures, some of them similar to terrestrial melanins or black particles. Under the action of radiation black particles explode, forming smaller fragments that are easier to identify. Black matter is in continuous evolution, producing low-molecular-weight organic molecules as it changes that eventually become ice. As an electrical conductor, black can regulate the chemistry and the balance of ions, radicals and molecules within interstellar clouds. The smallest fragments resulting from the explosion of black particles give rise to organic molecules similar or identical to some already known on earth : the concept of Organic Chemistry can be therefore extended to astrochemistry

The cosmochemistry of black matter may stimulate fresh interest on earth in research on the melanins, which so far has strayed along the wrong paths. Once we have straightened out problems of purification and extraction [17], further interesting developments can be expected in nanochemistry, nanobiochemistry and nanophysics of these black particles [5, 18-26].

Melanin is a conductor whose configuration varies under the action of electric or electromagnetic fields. Melanin assembles the simplest elements and can control the form and function of cell adhesion [18, 19]. Stellar melanin is a producer of organic molecules, while terrestrial melanin assembles organic molecules and macromolecules.

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In memory of the mathematician Donato Greco (1923-1995)

*It is not the only objective of organic chemistry
to elucidate the structure of organic natural
products but also to develop the nanochemistry
of Nature.*

Melanins,cosmoids,fullerenes.

Nota di Giancarlo Nicolaus e del socio Rodolfo A. Nicolaus

(Adunanza del 6 novembre 1999)

Key words : melanins,conducting polymers,charge transfer complexes,fullerene,graphite,
cosmoids, cosmochemistry.

Introduction

Black materials are present in the Biosphere (eumelanin,phaeomelanin,allomelanin), in the Lithosphere (minerals, carbon, graphite, fullerenes),in the Atmosphere (primary and secondary pollutants,smokes),in the Hydrosphere (pollutants) and in the Cosmos (interstellar black dust) (1,2).All black materials belong to the solid state,following its laws.The intra or extracellular black materials of the biosphere are called melanins,which are classified in three groups,the Eumelanins (produced by the polymerization of a nitrogenous melanogen),the pheomelanins (obtained from the polymerization of a sulphurated melanogen),the allomelanins (generated by the polymerization of polyphenols).Hybrids of the abovementioned groups can be easily formed through copolymerization or by the presence of foreign material.The polymerization of melanogens is of radical type. Attempts to isolate intermediates of relevant interest for the particles structure (melanins) failed completely.

The black pigments are solid materials with band structure having a gap less than 2eV. They are formed from melanogens which are organized in natural structures called melanosomes, having the soccer or rugby ball shape of fullerenes. The colours of melanins are those of a pure semiconductor. The natural materials show the properties of amorphous semiconductors with low electric activity at the borderline with an insulator state. As with graphite and fullerenes, this state is subject to changes by doping. All black pigments, melanin included, show a remarkable affinity for metals (presence of -COOH) as well as for organic electron donors. The biological properties of the melanin/chlorpromazine and melanin/quinine complexes have been well investigated. On the contrary, the ability of melanins to form charge transfer complexes has been poorly treated and limited data are available on their conductivity. Melanins after decarboxylation, or those arising from neutral melanogens, are able to bind acids and metalloids. In other words, there are melanins bound to counteranions (like pyrrole-black) as well as to counteranions. The conductivity of those materials is poorly understood. The oxidative cleavage of eumelanins yields nitrogenous polycarboxylic acids (like 2,3,4,5-pyrroletetracarboxylic acid), that of allomelanins graphitic acids (like mellitic acid), that of pheomelanins has not yet been sufficiently investigated. The presence of a graphitic core in melanins has been confirmed by X-ray analysis. A fullerene cage structure is proposed for black materials. A new approach to eumelanin research starts from an intact biological entity: the melanosome. In this way, physics and chemistry are referred to melanosomes for the first time. The laser beam represents in this connection a useful tool for their chemical investigation. It is worth mentioning that, under the action of a laser beam, both graphite and melanosomes are fragmented (typical for graphite is the formation of the red C₆₀ fragment). The fragments, which are formed through a reaction comparable to an explosion, can be easily detected and identified with MALDI-TOF, LDI mass spectrometry. In the common conditions which are used in the MALDI-TOF, LDI technique, melanins, as expected, are not volatile.

Black materials are wide spread in nature. Among the natural pigments melanins occupy a unique position thanks to their physical, biological and chemical properties. The most important physical properties are electrical conductivity in amorphous semiconductors (2), the display of "threshold switching", the existence of "planar stacks of monomer units

" ,the absorption of ultrasound in the region of 1 MHz , colour attributable more to the electronic transition in band materials than to orbital transitions and,finally,the capacity to form " charge transfer complexes " (drugs and metals affinity).In the biological field it is to be noted that mammals melanogenesis is a rare known radical process occurring *in vivo* .In the field of structural chemistry the research has given poor and confused results.

Biosphere

Melanins(1) have been the subject of numerous investigations, and a number have been isolated and subjected to chemical examination. A satisfying method for isolation and purification has not been found until today.The most satisfactory preparative method is carried out on the melanosomal fraction. The situation is less satisfactory among the lower animals where there is much reliance on inadequate histochemical tests for the purpose of identification, and much of the older work does little more than record the existence of melanins (black pigments, black materials) . Search is usually made for an accompanying tyrosinase system, although this too, is of uncertain value, as these enzymes are not specific.In consequence the results are ambiguous.Nevertheless it seems likely that the innumerable shades of black and brown frequently observed in vertebrates and invertebrates are produced mainly by melanins occurring in different states of oxidation and aggregation, accompanied sometimes by other pigments. As may be seen from the literature these pigments are of general occurrence in many phyla.

The most convincing description of melanin in a primitive animal is the plumose anemone *Metridium senile*. Coloured varieties were found to contain a black granular pigment in the endoderm which responded to the usual tests and a more diffuse brown melanin in the ectoderm; both white and coloured animals possess a complete tyrosinase system. Tyrosinase is present in the tissue fluids of sponges (e.g., *Suberites domuncula*, *Tethya lyncurium*), but its relevance to the black pigments occasionally observed in the Porifera is unknown. It is of interest that a magenta chromo-protein found in the jellyfish, *Pelagia noctiluca* contains a brown chromogen which appears to be an indole derivate. However, no tyrosinase could be detected, and its relationship to melanin, if any, is

doubtful. The flatworms *Planaria lugubris* and *Polycelis nigra* vary in color from white to black. The histochemical tests show, that the characteristic black granular pigment in the large epidermal melanophores of *Diadema antillarum* and *Thyone briareus* is melanin. The amebocytes of these animals are associated with a phenolase complex, and if the coelomic fluid is exposed to air, rapidly become red and then gradually darken to brown or black, producing a pigment very similar to that in the skin. A similar enzyme system is present in the amebocytes of the holothurioid echinoderm *Holothuria forskali*, which contains an abundance of melanin in the body wall. Integumentary black pigment is frequently visible in gastropods and bivalves, and a copious supply is present in the specialized anal gland or ink sac of cephalopod mollusks. The sac of *Sepia officinalis* contains up to 10 ml of an intensely black suspension of melanin granules. Tyrosinase activity in both *Sepia* gland and dried *Sepia* sac has been reported. There is a recognizable pattern in the colours of deep water fishes.

The animals that inhabit the surface down to about 150 m are either transparent or blue; below this depth and down to about 500 m the inhabitants are mainly silvery and greyish fish; below this depth again the population consists of dark coloured fishes (Fig. 1) and red prawns (*Taonius megalops*, *Calliteuthis reversa*, *Taonius megalope*, *Gnatheuphausis*, *Pentacrinus*, *Astronesthes*, *Melanocetus*). Many of the deep sea cephalopods such as *Mastigoteuthis* and *Calliteuthis* are also red. The development of the colour may have a concealing effect since at depths where there is no red light, these creatures will appear black. This colouration is peculiar to salt-water animals that dwell in dark regions whereas cave-dwelling animals are usually a pallid white. The colours of deep sea animals may, of course, have no ecological significance and they could act as stores of waste products or of metabolically useful intermediates. It is known that starvation causes the depletion of carotenoids from the ink and liver of *Octopus bimaculatus*. In animals such as the euphausiids their red colour renders them invisible in the twilight zone of mid-ocean waters where red light does not penetrate, and they are not rendered visible by the light usually emitted by predators. But some of the fishes that feed upon them emit red light from large luminous organs. It may be that the colours of some fishes have been influenced not only by incident light from above but also by the need to avoid reflections from light emitted by predators. The melanin structure of these fish is interesting because, as is

known, pigment formation is influenced by incident light, oxygen, and pressure. Unfortunately no research on the conductivity of these interesting pigments has been hitherto carried out.

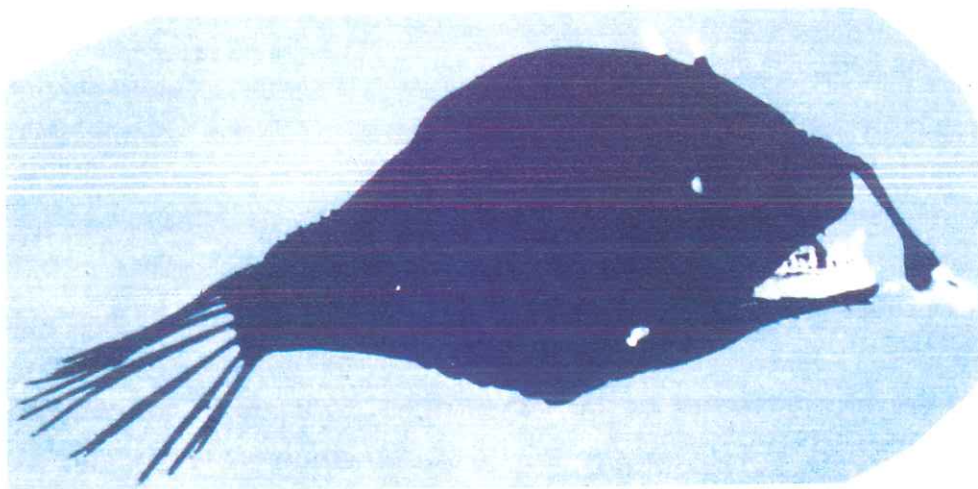


Fig. 1 Adapted from P.N.Dilly "The enigma of colouration and light emission in deep-sea animals" Endeavour XXXII, n° 115 (1973).

The black pigment in the hypodermis of certain crustaceans (e.g., *Cancer pagurus*, *Crangon vulgaris*) seems to be associated with tyrosinase and shows the general characteristics of melanin. There is good evidence that melanin formation is responsible for the undesirable "black spot" darkening of fresh shrimp. The blood of many insects, of which tyrosine is a common constituent, rapidly darkens on exposure to air; likewise, many pale newly hatched larvae darken in a short time after emergence. This is associated to a polyphenol system which is very widely distributed in insects and melanin formation is clearly implicated, although the pigment is not usually granular. The position is complicated, however, by the presence of protocatechuic acid and other polyphenols, and it has been difficult to disentangle the process of melanization from cuticular darkening due

to sclerotisation. However, the two processes are to some extent independent. In experiments with the cuticle of the desert locust (*Schistocerca gregaria*) it was found that blackening could occur unaccompanied by tanning. There is much evidence that tyrosinase is distributed generally throughout insect tissues, but a clear relationship between pigments and the presence of the so called tyrosinase has not yet been well established. The chordate melanins are frequently prominent in skin, hair, feathers, and scales and occur also in choroid, peritoneum, pia mater, and other membranous tissues in mammals, birds, reptiles, amphibians, and fishes. Melanotic tumours are not uncommon among vertebrates. The type of melanins and the size and shape of melanosome is genetically determined, but the radical process which forms melanin may be influenced by UV light, pH, O₂, pressure and other factors. Unknown biological factors cause pigmentary disorders like vitiligo, albinism, and melanoma (1e). Melanoma is not limited to humans but develops in horses and other groups of vertebrates including fish of the *Xiphophorus* genus (1f). The available evidence (EPR, "colour", electroactivity, etc.) suggests that mammalian melanins are probably all similar in nanochemical character. Less is known concerning the nature of other vertebrate melanins although their distribution in birds and amphibia has received much attention by geneticists, and some remarkable patterns of melanization have been described. Integumentary melanins are indirectly responsible for many structural colours displayed by animals, notably the blues due to Tyndall scattering seen in certain skin areas of numerous fishes, reptiles, and mammals and in the feathers of birds. Pigmentary and structural colours are sometimes combined, this can be observed in the green feathers of birds which change to Tyndall blue on extraction of the concomitant yellow carotenoids. Melanin also plays an important background role in the display of iridescent (interference) colours frequently seen, for example, in the skin of reptiles and fishes. The coexistence of black, blue, and green areas in many fishes, together with changeable iridescent colours, is not uncommon, and all these can be attributed, directly or indirectly, to the relative distribution of melanosomal pigments in melanophores.

Neuromelanin is a granular pigment of the nigrostriatal neurons in the brain stem of humans which in many respects is different from the melanin formed in epidermal melanocytes. The highest levels of neuromelanin are found in the neurons of the *Substantia nigra* and *Locus coeruleus* which are known to contain relatively

high concentrations of dopamine and norepinephrin. Small amounts of neuromelanin has been found in horses, carnivores, guinea pigs, rabbits, rats, mice and amphibians.

Neuromelanin occurs almost exclusively in catecholaminergic neurons (1d). Probably the most important property, from a biological point of view, of these pigments is the electroactivity.

The formation of dark-brown or black pigments during the normal development of plants is a well-known phenomenon commonly observed as markings on petals and leaves, in the spores and hyphae of the higher fungi, in senescent leaves and seedpods, and in the dead cells of bark, seedcoats, and pericarps. Furthermore, many plant tissues darken rapidly on injury. It is likely that many of these pigments arise by the oxidation of phenols followed by conversion into complex black products by polymerization and interaction with proteins and aminoacids.

The Japanese lacquers are the oxidation products of polyphenols exuded by many *Anacardiaceae*, and the black phytomelanes elaborated in the fruits of certain *Compositae*. It may be noted that the essential oils of many *Compositae* contain polyacetylenic compounds (which remember acetyleneblack) which also occur in higher fungi. These highly unsaturated substances frequently are unstable and blacken on exposure to light. The glucoside, aucubin, is responsible for the blackening of the leaves, fruits, and other parts of the Japanese variegated laurel *Aucuba japonica* and is found in the leaves and seeds of many other plants. The black material is an oxidation product of the aglycon aucubigenin.

Another example of an allomelanin is seen in the fungus *Daldinia concentrica* formed from a naphthalene polyphenol which is oxidized to a black material.

Black material is obtained from Dahlia tubers and can be detected in the broom *Sarothamnus scoparius*.

The blackening of bananas, which occurs frequently on storage, is due to oxidation of dopamine present in the skin and pulp. A number of bacteria (e.g., *Bacillus niger*) produce black pigments. The melanin (containing nitrogen) formed by one of these (*B. salmonicida* or a close relative) forms a dark brown solution in aqueous sodium hydroxide which was

reprecipitated by acid. Little comparable work appears to have been done on the presumed melanins of fungi or higher plants, although the blackening of potato tubers has received some attention. Much more work on the isolation and purification of plant melanins is required, as only in this way can they be properly identified and compared with the animal pigments. From *Vicia faba* dopa may be extracted ;a hint as to the possible course of melanogenesis is given by the co-occurrence of tyrosine and dopa in a number of melanin-producing plants and by the observation that the blackening of plant tissue sometimes proceeds via a red phase, as, for example, when potato slices are ground up or exposed to chloroform vapor. Very interesting is the mould *Aspergillus niger* which produces Aspergillin, a graphite- like material (2) (8) .Fig.2



Fig.2.The photo was a gift of Professor A.Quilico (Milan 1960). *Aspergillus niger* an organism that generates a rare graphite-like pigment.

Moreover,black pigmentation is often highly localized in individual plants in contrast to the more general distribution of the enzyme "tyrosinase". This may be attributed either to the presence of inhibitors (ascorbic acid is of common occurrence) or to circumstances which, under normal conditions, prevent the enzyme and substrate having simultaneous access to oxygen and more probably to the fact that the final stages of melanogenesis does not require the presence of an enzyme . This situation breaks down on disruption of the cell structure and melanin formation can then proceed; alternatively injury to the tissue may result in oxidation of the inhibitor with consequent loss of inhibitory capacity. An example

of ascorbic acid inhibition is seen in *Stizolobium hassjoo*; blackening takes place most rapidly in the younger leaves, which have the lowest concentration of ascorbic acid. A study of melanin formation by a black mutant of the fungus *Neurospora crassa* it was possible to correlate increased pigmentation with increased tyrosinase activity, but was unable to demonstrate whether this was due to the presence of a larger amount of enzyme or to a partial block in the synthesis of a tyrosinase inhibitor. The inhibitor in this case may be a thiol, as it was found that tyrosinase activity in *Neurospora* is dependent upon sulphur nutrition. The fungal pigments are in general wallbound and extracellular in nature and are formed by a radical process starting from diphenols (catechol, 1,8-dihydroxynaphthalene, glutamyl-3,4,-dihydroxybenzene). To date the only pigments fully characterized by chemists are those of the large ascomycetes *Daldinia concentrica*, *Ustilago maydis*, and *Aspergillus niger*. *Ustilago maydis*, a well-known corn parasite (Fig. 3), was found to have a pigment which is a polymerized polycondensed product of catechol or its derivatives (1c), (1d).



Fig.3 .The corn parasite *Ustilago maydis* which causes serious damage to agriculture(Institute of Organic Chemistry,University of Naples,1962).

Black materials

Black materials are found not only in the biosphere but also in the lithosphere (minerals,humic acids,graphite,blackshale),in the hydrosphere (black particles in seas,lakes,rivers, originating probably from the biosphere) ,in the atmosphere (pollutants,soot) , and the cosmos (cosomids,graphite).

Many synthetic pigments (materials) are also known ,some of them related to melanin such as :

pyrroleblack, indoleblack, benzeneblack, quinolineblack, DHIblack, adrenalinblack, tryptophanblack, serotoninblack, DOPAbblack, acetyleneblack, dihydroxyquinolineblack catecholblack, polyphenyleneblack .

In the last fifty years melanins have been intensively studied from a chemical and physical point of view without great success. Some precursors (melanogens) have been discovered but the process of melanogenesis and its biological significance remain to a large degree obscure. Although in recent years , many organic compounds and many black polymers were found to be conducting materials, the electroactivity of melanins has never seriously been taken into consideration. Consequently the biological aspects of the problem remained confused.

Chemistry and Physics

The black pigments are band structure materials with a small gap ($eV = 0.1 - 1.7$) value, as expected from theoretical calculations.

Generally they are amorphous semiconductors but black crystalline charge transfer complexes are also known. Melanins behave like amorphous semiconductors their colours being generated more by band transitions than by electronic intramolecular transitions as usually reported for natural and synthetic pigments (2),(3).

Melanins, the natural black pigments, belong to the solid state, or to the nanostructure area which comprises supramolecular chemistry which studies the chemical properties of aggregates, particles, objects of 10-100 nm. dimension ($1 \text{ nm} = 10^{-9} \text{ m}$; $1 \text{ nm} = 10 \text{ \AA}$). This concept of supramolecular or nanostructural chemistry was never applied to melanins. Nanochemistry requires, particles having almost one characteristic dimension lying between 1 and 50 nm. An approach to melanin nanophysics, nanobiophysics, nanochemistry is tentatively described.

Melanogens are compounds which in different chemical and physical conditions produce black materials usually in aggregates or particle form. In nature these particles are organized to structures called melanosome with the characteristic shape of a soccer or rugby ball.

The colours of "melanins" correspond to the colours of the pure semiconductors of Fermi's prohibited band. (Fig.4) Many black materials are found in nature and are



Fig.4 Cadmium sulphide has extension of prohibited Fermi's band of 2.6 eV and for this reason appears yellow, whereas cadmium selenide with a value of 1.7 eV is black. The reddish colours are intermediates. Adapted from F.Celentano "Luce colore e materia", Le Scienze, quaderno 21, Febbraio 1985.

called eumelanin, pheomelanin, allomelanin.

They are electroactive materials (2,9). The conductivity depends on :

- a) the type of counteranion or countercation
- b) the method and conditions of preparation
- c) the type (orientation) of conductivity measurement adapted: parallel or orthogonal to the plane of basal cleavage
- d) the number of unpaired electrons.

Synthetic or biosynthetic melanins for conductivity experiments may be prepared by electro-oxidation of melanogens. Alternatively the preparation of a melanin can be carried out with very small quantities of peroxidase/ H_2O_2 . The yield of melanin is a good criterion for evaluating whether the oxidative method chosen is suitable. The preparation which brought about the discovery of the pyrroleblack conductor (3) is still valid to day. The polymerization is carried out by electrodes (duration 2h) with a constant current of 100 mA of a melanogen solution (2g) in 100 ml 0.1 N H_2SO_4 . The laminar deposit which forms on the platinum electrode is washed out with distilled water and allowed to dry. The conductivity is measured both on the plaque and on pills obtained from the powdered plaque. The blacks give an EPR signal in the range of 2,0025 – 2,0045 g values. The blacks have the electrolyte of the medium incorporated as a counteranion. The observed values are reported in $\sigma = \Omega^{-1} \text{ cm}^{-1}$. The films only form if the monomer is oxidised from above its oxidation potential. If the film which is deposited is a conductor the film continues to grow and the current continues to pass. The pigments are described by shade colours as black, blue-black, matt-black, brilliant-black, pitch-black, brown-black, copper-black, brilliant-black, golden-black, metallic-black. Melanins and black pigments show an affinity for metals dependent to a large extent on presence of $-\text{COOH}$ and for organic electron donors. Biologically the complex chlorpromazine/melanin and quinine/melanin is well studied. The possibility that melanins could form charge transfer complexes has been poorly considered. In consequence there is little information available on the conductivity of these complexes. If melanin is decarboxylated or formed from a neutral precursor the polymer may bind to acids and metalloids.

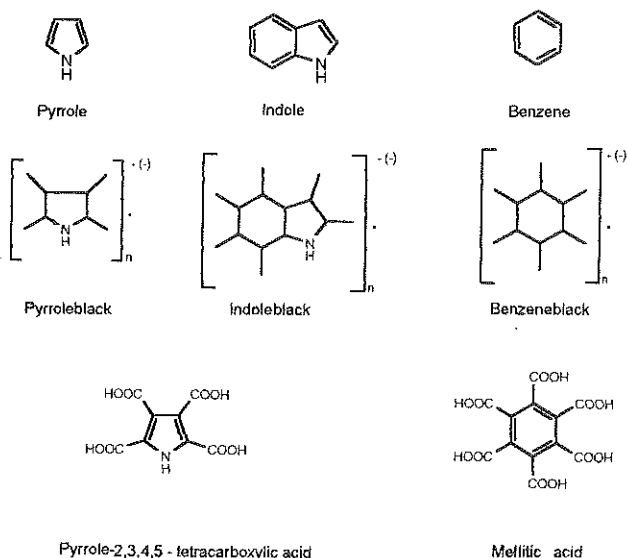
Black pigments have the characteristic properties of amorphous semiconductors; that is weak electrical activity ($10^{-11} - 10^{-7} \Omega^{-1} \text{ cm}^{-1}$) at the limit of the insulating state, the conductivity being increased to interesting value by doping.

Actually the chemistry of melanins is still obscure. The causes are due to :

- a. having considered the transformation of the melanogen into melanin a process under enzymatic control.
- b. having worked with altered or not purified materials.
- c. having not understood the very nature and electroactivity of black materials and of melanins .

The re- examination of old papers and the extension of nanochemistry to the study of black materials has led us to a new aspect of melanins (2).

The pigments are classified as from three basic structures based on pyrrole, indole, benzene.



The formula above reported are called cetoplasmatic (2). They show the presence of unpaired electrons and cationic(anionic) centers with counterion. All the positions of the ring may be involved in polymerization. Oxidative level and polymerization degree are not known.

Pyrrole-black, indole-black, benzene-black and derivatives give, on oxidation, a mixture of polycarboxylic acids (4). The formation of these acids means that polycondensed or polysubstituted structures are present in those black pigments. Recently [see (2) "Coloured organic semiconductors : melanins " pag.34] a structure for an indole oligomer which can explain the formation of pyrroletetracarboxylic acid by oxidative degradation was reported . Pyrrole –2,3,4,5 – tetracarboxylic and tricarboxylic acids were extracted from the mixture of degradation products obtained on oxidation of melanins prepared from rat melanoma, human hair, dog hair, horse hair, ox hair, ox choroide, sepia ink(7), squid ink, octopus ink, chicken feathers, pigeon feathers, amphiuma liver, axolotl liver (4), sepia tapetum lucidum (5). The same tetracarboxylic acid was obtained from oxidation of pyrrole black and DHI (5,6 - dihydroxyindole) black. (5,5b)

2,3,4,5 – Pyrroletetracarboxylic acid is a characteristic product which is found in the oxidative degradation mixture of eumelanin. It was first prepared in 1954 (7) From a structural point of view this acid is the most significant product yet found .It shows the presence of a "graphitic core" in the eumelanins.

Pyrrole – 2,3,4,5 – tetracarboxylic acid combines with the ion K^+ to form a monopotassium salt which is insoluble in strong acids, in water and in organic solvents . It is believed to be a "clathrate". In order to readily isolate and identify the acid it is necessary to oxidize the pigment with peracetic acid (1b) avoiding any contact with sodium or potassium. In these conditions the yield is close to that of 2,3,5 –pyrrole-tricarboxylic acid and considerably higher to that of the dicarboxylic acids . Curiously, some authors who have been unable to isolate or identify the acid claim that it is not a genuine product, i.e. it is an artefact (5) . Recently an interesting polysubstituted indole structure has been proposed for an indole tetramer which could be the terminal part of indole-black (the simplest eumelanin known). The structure presented (6) justifies the formation of the acid pyrroletetracarboxylic in the oxidative degradation processes.

In conclusion, every analytic and structural approach to the study of eumelanins which does not consider the presence of the pyrroletetracarboxylic acid among the degradation products must be considered invalid.

The planar graphitic (or giant fullerene) core present in melanins may show remarkable anisotropic properties. The value of physical quantity depends on the considered

direction since both electric and thermic conductivity are higher if measured parallelly to the cleavage plane (9).

For physical and chemical properties graphite may be considered the most simple allomelanin. The formation of mellitic acid on oxidation is a feature common to some allomelanin in particular Aspergillin (8).

Furthermore graphite and allomelanins have in common the colour, the EPR signal the X-ray diffraction pattern, and electroactivity. No vaporization of melanin occurs in the usual condition of the MALDI-TOF (matrix assisted laser desorption ionization-time of flight) or LDI techniques, but an explosion is produced by the action of laser on melanosomes of *Sepia* ink (9). This is a confirmation of what occurs with epidermal melanosomes (15) and the result probably may possibly be extended to all melanins.

The allomelanins which probably incorporated graphite structures must include *Daldinia-melanin*, *Ustilago-melanin*, Humic acids, and Aspergillin (10) (11) (12) (13).

This latter material is particularly interesting because oxidative degradation produces mellitic acid the same acid which can be obtained from graphite. In other words it may be suggested that aspergillin is a "substituted graphite" synthesized by a living organism.

Aspergillin (8) is a pigment which gives to *Aspergillus niger* its characteristic dark colour. (Fig.2) The typical colouration of the conidia is one fundamental distinctive character of numerous species. The appearance of the pigment in the spores is at first yellow-ish, becoming green-yellow, green-grey, and finally brown-black.

These "colours" are typical of amorphous semiconductors. The oxygen in this synthesis plays a special role next to the iron in that the quantization of the chemical elements can lead to a gold-yellow pigment. The IR spectrum in the molecular phase is very similar to that of humic acids. Band theory may be extended to these amorphous semiconductors with polycondensed or polysubstituted nuclei.

The pigment is purified by dissolving it in NH_4OH 5% (solubility 1 g per 1000 ml) and reprecipitating with 2N HCl and washing with H_2O . The dissolving in NH_4OH and the precipitation with HCl are carried out several times. The insoluble fractions are discarded. The final product is washed with H_2O , ethanol, acetone, H_2O . The wet product is dissolved

in water and the solution is used for measuring the conductivity (difference between conductivity of the solution and of distilled water) or is studied in the solid state.

The elementary analysis (Sephadex purification) of aspergillin gives C%=52,7 H%=4,0 O% 36,0 . Nitrogen which is present may be part of a protein (9). Aspergillin oxidised with H_2O_2 10% yields mellitic acid (1g from 4 g pigment) and oxalic acid; on reduction with Raney Ni it yields perylene next to hydrogenated derivatives of phenanthrene and naphthalene.

Aspergillin can be dissolved both for measurement of the electrical conductivity (perylene itself is a good conductor if doped) and for spectrometry studies.

Further indications of a polysubstituted or polycondensed structure of melanin derives from X- ray studies.

A number of " purified "natural and synthetic melanins have been examined by X- ray diffraction (14). A diffuse ring, centered at a Bragg spacing of $3,4 \text{ \AA}^\circ$ was consistently found in samples of melanin from animal sources, and a similar ring at $4,2 \text{ \AA}^\circ$ in all melanins obtained from plants. Models for these two polymer types, based upon the current concept that they primarily involve indole and catechol monomeric units respectively, were then evaluated by a Montecarlo method. From the comparison of the observed spacings with the calculated ones it was concluded that the $4.2 \text{ \AA}^\circ$ spacing in catechol melanins is probably related to the average interaction between adjacent monomeric units with mutually random orientations. The $3.4 \text{ \AA}^\circ$ spacing observed in indole melanins appears to derive from the tendency of indole monomers (probably of adjacent chains) to aggregate in near parallel stacks. Some randomness in the form of translations and relations parallel to the planar groups is consistent with diffraction patterns. An interesting finding was that the diffraction pattern of synthetic of catechol melanins gives the $3.4 \text{ \AA}^\circ$ spacing found in the indole melanins of animal origin. The nanochemistry of these interesting materials has not yet been further developed.

The X- ray diffraction data allow to conclude that there is a short range order at the molecular level in all melanins .A graphitic core is always present in eumelanin and allomelanin. A consistent finding with all samples was the lack of structure in their diffraction patterns of elements showing any significant crystallinity.

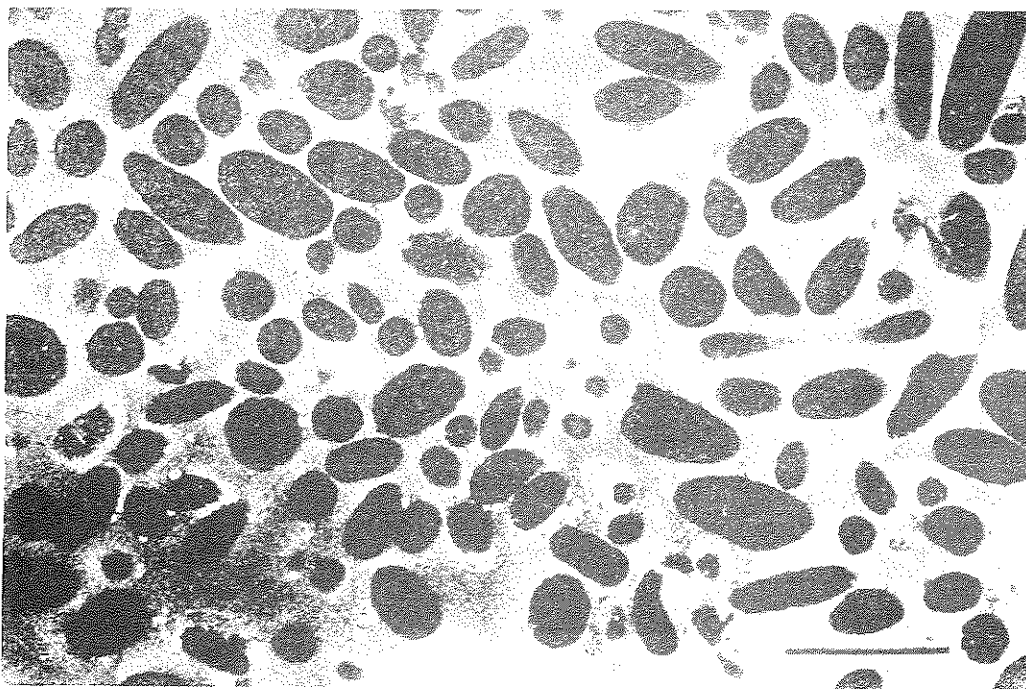


Fig.5. Electron micrograph of melanosome fraction. Taken from V.J Hearing, M.A Lutzner " Mammalian Melanosomal Proteins : Characterization by polyacrylamide Gel Electrophoresis " Yale J.Biol.Med. **46**,553 (1973)

The separation between adjacent planar groups in the plant and fungal melanins seems to be much larger than in the case of the animal and synthetic melanins studied. Density measurements on the melanin pellets seem to support this conclusion. The adjacent planar groups in the synthetic and animal melanins are probably parallel and they may also have order extending to longer ranges. The electronic properties of these melanins, including their black colour, may be related to the possible presence of "graphitic like structures" and to the shape and size of granules of the pigment as seen at the electron microscope level. The high density of these melanins may be responsible for the observed contrast in the electron micrographs of melanin – containing tissues as depicted in (Fig. 5).

The melanosomes show soccer and rugby ball shapes like those seen in electron micrographs of fullerenes. In the case of melanosomes the cage may enclose firmly proteins, lipids, metals melanogens, oligomers, water, gas.

Fullerenes are formed from fine aggregates of even numbers of carbon atoms and are obtained after evaporation and cooling of graphite (laser method).

It is possible to obtain red and black fullerenes (C_{60} magenta to C_{540} black). The C_{60} which can be obtained in relative good yield, forms a spheroid structure (icosahedron trunk). The sphere would be formed by the collapse and curling up of graphite. Fullerenes show interesting optical properties as non-linear variation of transparency, as also shown in melanin, as a function of intensity of incident light. In order to understand the dynamic aspects of fullerene formation a new model, the Wakabayashi-Achiba model, was suggested for the growth mechanism of fullerene (17). The model is based on the criteria that the fullerene is formed by a sequential stacking with numbers and combinations of even-numbered carbon rings (atoms) and both intermediates and the final product consist of only pentagons (cyclopentane, pyrroles) and hexagons (benzene) or a combined system (benzocyclopentane, benzofurane, benzopyrrole, indene). Although the process which leads to cage formation seems to require high temperature, a similar process occurring for *in vivo* melanosome formation cannot be excluded, especially for substituted hydroxyl hexagons. Recently formation of fullerenes by pulsed laser irradiation of gaseous benzene was observed (34).

A phenomenon very similar to the graphite-fullerene (giant fullerene) laser collapse occurs with melanosomes both *in vitro* and *in vivo* in the form of the explosive vaporization of melanosome. We suggest that melanosomes may be considered substituted ($H, N, OH, COOH$) fullerenic cage structures. Melanosomes and fullerenes may represent evolution phases. Intermediary of particles produced by stars and on the earth. Melanosomes are subcellular particles which contain melanin and a primary protein matrix. The complex melanin+protein is sometimes not chemically bound as in the ink sac. The black material which loses CO_2 by heating and by acid hydrolysis is, from a chemical point of view, very reactive. On the other hand electron micrographic studies show that melanosomes after strong acid hydrolysis preserved their size, shape and, to a great

extent, their ultrastructural features of either lamellar or granular melanosomes(23). Molecular weight, chemical structure can not be determined for melanins or melanosomes. Only the techniques of nanochemistry and nanobiochemistry could reach some success.

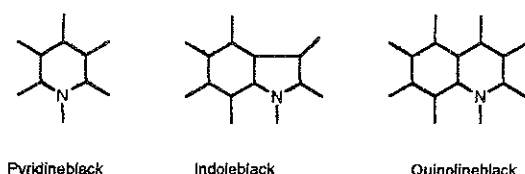
Recently the problem to characterize the black materials (melanins) by means of light scattering techniques, (static and dynamic was) investigated (32). The static technique allow to identify the "macromolecular" properties of the ink sac melanin, autooxidative dopamelanin, "enzymatic" dopamelanin. Natural melanin at physiological pH polymerizes and aggregates in particles of average molecular weight $\sim 10^6$ and radius of gyration R_g about 90 nm, with spherical symmetry and with an open, rigid and not largely hydrated texture. Synthetic melanin prepared by autooxidation at physiological pH appears the more reliable structural model for the natural material. Both natural and synthetic melanins grow as fractal aggregates of small units, with kinetics depending on the conditions of the medium. The black solid is considered in general a very stable material but melanin is easily disintegrated by lysosomal digestion, bland action of H_2O_2 , heating (loss of CO_2) and by LASER beam. This seems in agreement with a cage or globular structure of melanins. Pulsed lasers are capable of selective photodermal destruction of melanosomes (15).

When sufficient radiant energy is delivered from a pulsed laser to the skin, the melanosomes in the stratum *corneum* reach a temperature that exceeds the threshold for explosive vaporization(15). The result is an explosive event which disrupts the laminated stratum corneum structure to yield an observable whitening of the skin surface. Probably the explosion begins with dilatation of the metallo protein of the cage.

Fullerenes, cosmoids, melanins.

It was observed that some of the dust clouds of the Milky Way Galaxy were made of long carbon chain molecules (the polyynes). Among the polyynes molecules like HC_5N , HC_7N , HC_9N were discovered (16) (17) (18) (19).

On earth the atomic relationship of some of those polyyne structures corresponds to pyridineblack, and quinolineblack (two isomers). The question of the existence of indoleblack ($(HC_8N)_n$) and pyrroleblack ($(HC_4N)_n$) will be discussed later.



We have several different types of intermediates participating in the formation of carbon microparticles in the gas phase. Cumulene, polyyne chains, monocyclic ring have been suggested to occur. Some of the more stable clusters are closed cage fullerenes like C_{60} or other more reactive which ultimately result in characteristic carbon microparticles very similar to melanosomes. The small carbon species C_2 and C_3 were identified in comets and C_5 in the cool carbon star IRC+10216. In the positive ion mass spectrum of the clusters with up to 30 or so atoms the 11,15,19,23 magic number sequence ($n=4$) has been recognized. Giant fullerenes might also exist and would have icosahedral shapes. C_{540} would have a diameter three times that of C_{60} . The icosahedral shape is quite clear at this size. The structure is really truncated but the truncation is of microscopic dimensions. The surface is that of a smoothly curving net with more or less flat triangular surfaces between the 12 cusps (18,19)

These observations represent a surprising and important breakthrough in our knowledge of the carbon content of space and the biochemistry of the earth. Further work drew attention to red giant stars which were pumping vast quantities of carbon molecules, as well as carbon dust, into space and suggested that there might be a link between carbon chains and soot formation (17).

The possible relationship between melanins and the structural features of giant fullerenes is interesting from a point of view of organic chemistry and the science of the

solid state. Considering the chemical and physical relationship existing between melanins (in their ordered part) and giant fullerene a cage structure may be present in the black materials in nature.

Nanochemistry has been largely applied to the study of fullerenes. It was found that crystalline C_{60} undergoes a phase transition to a simple cubic structure at 260 K which is accompanied by an abrupt lattice contraction. Further approach to fullerenes structure has been made with the help of geometry (16,17)

C_{60} was predicted to be a possible by product of soot formation and subsequently shown to occur in a sooting flame. Giant fullerenes were shown to possess quasi-icosahedral structure consistent with the polyhedral shapes of certain graphite microparticles (18).

Powerful mass spectrometric techniques for studying refractory clusters generated in a plasma by focusing a pulsed laser was developed, a technique which offered a way of simulating the chemistry in a carbon star if a graphite target was used.

The most significant observations to emerge were:

- a) Atoms could be encapsulated in the fullerene cage
- b) Giant fullerenes possess quasi icosahedral structure consistent with the polyhedral structure and shapes of certain melanosomal microparticles.
- c) The transition temperature are sensitive functions of both the crystallinity and the purity of the solid fullerene [17-19].

On the contrary of melanins the properties of doped fullerenes have attracted much attention since the first detection of these molecules. At first, fullerenes with metal inside were produced by laser vaporization of metal mixed with graphite, and endohedral fullerene complexes with lanthanum, potassium and cesium were formed. Later the superconductivity of C_{60} doped with rubidium, cesium and other alkali metal alloys stimulated interest in the metallic and superconductive properties of fullerides with alkali or alkaline earth metals. The most surprising result is the production of CO^+ and CO_2^+ masses, for both the undoped and doped fullerene targets when the number of laser blastings is increased. For the target containing lanthanum oxide it may be supposed that CO^+ and CO_2^+ species are produced by dissociation of lanthanum oxide by a photochemical reaction of the uv radiation on fullerene. For the undoped fullerene target

the CO^+ and CO_2^+ production should be associated with the release of O_2 adsorbed in the bulk, photochemically reacting with C_{60} .

Infrared, Raman and uv spectroscopic analyses have been performed on thin films formed on KBr and quartz substrates by the laser ablation of C_{60} and C_{60} mixed with La_2O_3 targets. The results of the spectroscopic investigation performed so far, are not very satisfactory because no infrared or uv bands associated to C_{60} could be clearly identified. Laser acts on graphite producing a series of degradation products whereas, in the case of immature melanosome, an explosion occurs both in vitro or in vivo with the formation of small molecules (9). The fragmentation of the melanosomal cage resembles the breaking of crockery. Among the various fullerenes the more stable is C_{60} . May be present in form of C^+ , albeit in minute amounts, in a sooty Bunsen burner flame, and among the combustion products of cars deposited on a tunnel wall. The pinkish red fraction isolated from the tunnel of Victory in Naples in 1964 may, in retrospect, be attributed to a fullerene fraction (20,21). Recently fullerenes were found in the fossil of a dinosaur egg (22). The fullerenes have been experimentally and theoretically known for long time and a large number of papers have been devoted to this subject (25). Many attempts have been made at rationalizing the electronic structure of the fullerenes, in particular to understand how the mixing of five- and six-membered rings in a three-dimensional structure affects the properties of these molecules. The behaviour of the fullerenes in electric fields, although direct measurement of conductivity were not made, and in particular their nonlinear properties, attracted much early attention due to the potential use of this class of "molecules" as efficient non linear optical devices. Interesting is the first degenerated four-wave mixing measurements on C_{60} which later was shown to be three orders of magnitude too large (24). Later studies indicated that the potential of fullerenes as efficient materials in non linear optics can be at best be considered modest. The presence of five- and six-membered rings in the fullerenes has prompted a number of studies of the magnetic properties of these "molecules" as the prototype for spherical aromaticity, a concept which may be enlarged to include melanins.

The synthesis of oxide such $[\text{C}_{120}]\text{O}$ was described (28). The dimeric fullerene oxide is the precursor for the synthesis of odd-numbered fullerenes. A new hydrothermally method of extraction (28) produces oxide such $[\text{C}_{120}]\text{O}$ and $[\text{C}_{130}]\text{O}$ and several dimeric fullerene

oxide were obtained in small amount with higher fullerene oxides. The generation of semiconducting polymers has been achieved by incorporation into host matrices formed by conventional polymers such as polyethylene and polystyrene (29). Similar work with melanosomes and melanins has not been carried out. The utilization of the strong photoluminescence of conjugated polymers for light emitting devices resulted in the emphasis on research into this class of materials that combine the electronic and optical properties as well as processing properties of polymers. The characterization of C_{60} as an electron acceptor capable of accepting as many as six electrons candidates as acceptor in blends with conjugated polymers as good photoexcited electrons donors. A wide class of these conjugated polymers and oligomers (the binding affinity of melanins) shows a photoinduced electron transfer from the excited state of the conjugated polymer onto the fullerene C_{60} . The stabilization of the charge separated state in the conjugated polymer is assumed to result from the stability and delocalization of the positive polaron on the conjugated polymer backbone. A simple, new, carbon nucleation scheme has been developed which results in quasi-single crystal particles of concentric, spiral-shell internal structure and overall quasi-icosahedral shape. Intermediates consisting of curved graphitic networks and overall growth controlled by epitaxy are key factors in the scheme which also produces C_{60} as an inert close cage (31). The scheme which explains the occurrence of polyhedral carbonaceous particles may also apply to soot and circumstellar dust formation. Giant fullerene may be useful in the nanochemistry and nanophysics studies of melanins in particular and of black material in general.

Electro-oxidation of L-DOPA was recently applied to prepare melanin and the effect of the electrolyte ions on the rate of charge propagation in the polymer was investigated (33). The decrease of the rate of charge transport observed upon increasing of electrolyte concentration may be explained by the distance between adjacent electroactive sites due to a swelling of the polymer film which reduces electron hopping between sites. Spectroelectrochemical data show that in melanin electroactive quinones may be present. The charge transport properties of this heterogenous material has been shown to compare very well with those of other conducting "polymers". Consequently DOPA melanin may be used as a matrix for preparing reagentless enzyme-based biosensors (33).

Taking in account all that is discussed above we conclude that the giant fullerene-graphite system present a model, which may stimulate the study of melanins and emphasizes the value of nanophysics as tool to inquire into the properties of melanosome.

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TEMPERATURE DEPENDENT VISCOSITY AND ITS INFLUENCE ON THE ONSET OF CONVECTION IN A POROUS MEDIUM

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Abstract A porous medium saturated by a fluid with exponentially temperature depending viscosity is considered. Conditions ensuring nonlinear stability in the L^2 -norm are obtained for the Darcy model.

Riassunto. Si considera un mezzo poroso saturato mediante un fluido con viscosità dipendente esponenzialmente dalla temperatura. Si ottengono condizioni assicuranti la stabilità non lineare nella norma L^2 per il modello di Darcy.

1 - Introduction

In the theory of nonlinear energy stability in fluid mechanics [22, 8, 19, 21, 24, 5] in order to recover results close to that obtained by linear instability theory, "generalized" energies (i.e. Liapunov functions different from the usual L^2 -norm) have been introduced [9, 14, 16, 19-21, 24]. But the use of generalized energies has a serious drawback. In fact often it happens that as close is the parameter threshold of nonlinear stability to that obtained by linear instability theory as a major restriction has to be made in the size of initial data. Precisely, if R_E denotes the nonlinear stability threshold and R is the parameter such as Rayleigh number then typically $E(0) \leq \alpha(R_E - R)^\beta$, $\alpha > 0$, $\beta > 0$, where $E(t)$ is the (generalized) energy, and hence as $R \rightarrow R_E$, $E(0) \rightarrow 0$, i.e. the radius of attraction becomes vanishingly small. This is a particularly serious restriction

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expecially when $R_E < R_L$, where R_L is the linear instability threshold. When $R_E = R_L$, it is more acceptable that the initial data has to run to zero when $R \rightarrow R_L$. In recent time many efforts have been devoted to achieve nonlinear stability results close to that obtained by linear instability theory without too restrictive conditions on the initial data [10, 11].

In this paper, for the convective stability in a porous medium saturated by a fluid with temperature depending viscosity, we achieve a nonlinear stability result in the L^2 -norm. The nonlinear stability threshold – less than which one of the linear instability – is obtained without too restrictive conditions on the initial data and the radius of attraction does not become vanishingly small.

We obtain the nonlinear stability result by assuming that:

- i) the initial condition of the perturbation $\theta(P, t)$ to the temperature field belongs to the Sobolev space $W^{2,2}(\Omega)$;
- ii) the most appropriate (at least for the liquids [3, 7, 18, 23, 26]) constitutive law for the dynamic viscosity $\mu(T)$ is²

$$(1.1) \quad \mu(T) = \mu_0 \exp[-\gamma(T - T_0)], \quad \gamma = \text{constant}(> 0)$$

where T is the temperature field, T_0 is a reference temperature and $\mu_0 (> 0)$ is the dynamic viscosity at temperature T_0 ;

- iii) the equation governing the flow in the porous medium is the Darcy equation.

The plan of the paper is the following. In Section 2 we introduce the problem and the perturbation equations to the conduction solution. Then, after having obtained an a priori estimate of the perturbed field to the temperature (Section 3), in Section 4 we carry out the linear stability analysis through the energy method and observe that the condition obtained is not only a sufficient condition but also a necessary condition. In Section 5 we present the table of the critical Rayleigh-Darcy number for the linear stability analysis. Finally in Section 6 we perform the nonlinear stability analysis of the conduction solution in the L^2 -norm and determine a nonlinear stability threshold.

²In the sequel we shall call “exponential fluids” the fluids for which the viscosity depends on the temperature through (1.1).

2 - Statement of the problem

Let us consider an infinite horizontal layer saturated with a homogeneous newtonian fluid under the action of a vertical gravity field $\mathbf{g} = -g \mathbf{k}$ in which an adverse temperature gradient is maintained. The fluid is contained in a porous medium between the planes $z = 0$ and $z = d$ with assigned temperatures $T(z = 0) = T_1 + \Delta T$ and $T(z = d) = T_1$, where $\Delta T > 0$ is the (positive) temperature difference between the bottom and the top planes of the layer.

Taking into account the law (1.1) for the dynamic viscosity, the Darcy - Oberbeque - Boussinesq (DOB) equations are [2, 8, 12]:

$$(2.1) \quad \begin{cases} \nabla p = -\frac{\mu_0}{k} \exp[-\gamma(T - T_0)] \mathbf{v} + g \rho_0 \alpha (T - T_0) \mathbf{k} \\ \nabla \cdot \mathbf{v} = 0 \\ AT_t + \mathbf{v} \cdot \nabla T = k_1 \Delta T, \end{cases}$$

where: T is the temperature, $A = \frac{(\rho_0 c_0)_m}{(\rho_0 c_0)_f} (> 0)$ is a dimensionless parameter in which the subscript m and f refer to the fluid-solid mixture and to the fluid respectively, $\mathbf{k}$ is the unit vector of the upward vertical, $\mathbf{v} = (v_x, v_y, v_z)$ is the seepage velocity field, ρ_0 and μ_0 are respectively the density and the dynamic viscosity of the fluid at temperature T_0 , p is the reduced pressure, g is the gravity, α is the thermal expansion coefficient, k is the permeability, c_0 is the specific heat at constant pressure, $k_1 = \frac{k_m}{(\rho_0 c_0)_f}$ is a dimensionless parameter in which k_m is the overall thermal conductivity.

To the system (2.1) we add the boundary conditions

$$(2.2) \quad T(z = 0) = T_1 + \Delta T, \quad T(z = d) = T_1,$$

with $\Delta T > 0$. On introducing the following dimensionless quantities:

$$t = t^* \frac{d^2}{k_1}, \quad p = p^* \frac{\mu_0 k_1}{k}, \quad \mathbf{x} = \mathbf{x}^* d, \quad \mathbf{v} = \mathbf{v}^* \frac{k_1}{d}, \quad T = T^* \Delta T,$$

$$R = \sqrt{\frac{\alpha g \Delta \tau d k \rho_0}{k_1 \mu_0}}, \Gamma = \gamma \Delta \tau, T_0 = \frac{T_0^*}{\Delta \tau}, T_1 = \frac{T_1^*}{\Delta \tau},$$

where, in particular, R is the Rayleigh number for the DOB equations, dropping all asterisks, the dimensionless form of the system (2.1) in the strip $\mathbb{R}^2 \times [0, 1]$ is:

$$(2.3) \quad \begin{cases} \nabla p = -\exp[-\Gamma(T - T_0)]\mathbf{v} + R^2(T - T_0)\mathbf{k} \\ \nabla \cdot \mathbf{v} = 0 \\ AT_t + \mathbf{v} \cdot \nabla T = \Delta T, \end{cases}$$

to which we append the boundary conditions:

$$(2.4) \quad T(z = 0) = T_1 + 1, \quad T(z = 1) = T_1.$$

Here we study the stability of nonconvecting stationary solution $m_0 = (\bar{\mathbf{v}} = 0, \bar{T}(z) = -z + 1 + T_1, \bar{p}(z))$, where $\bar{p}(z)$ is such that

$$\frac{d\bar{p}(z)}{dz} = R^2(\bar{T}(z) - T_0).$$

On denoting by $R\mathbf{u}, \theta, \pi$, respectively, the dimensionless perturbation to the velocity, temperature and pressure fields with $\mathbf{u} = (u, v, w)$, the dimensionless equations governing the evolution equations of the perturbation fields in the strip $\mathbb{R}^2 \times [0, 1]$ are:

$$(2.5) \quad \begin{cases} \nabla \pi = -R \exp[\Gamma(z - \xi) - \Gamma\theta]\mathbf{u} + R^2 \theta \mathbf{k} \\ \nabla \cdot \mathbf{u} = 0 \\ A\theta_t = \Delta \theta + R w - R \mathbf{u} \cdot \nabla \theta, \end{cases}$$

with

$$(2.6) \quad \xi = T_1 + 1 - T_0.$$

To the previous system we add the boundary conditions:

$$(2.7) \quad w = \theta = 0 \quad \text{on } z = 0, 1,$$

and the following initial conditions:

$$(2.8) \quad \mathbf{u}(P, 0) = \mathbf{u}_0(P), \quad \theta(P, 0) = \theta_0(P).$$

In the sequel we shall assume that the perturbation fields $\mathbf{u}$, θ and π are sufficiently smooth and that they are periodic functions in the x and y directions, of periods $2\pi/a_1$, $2\pi/a_2$. We shall denote by $\Omega = [0, 2\pi/a_1] \times [0, 2\pi/a_2] \times [0, 1]$ the periodicity cell, by $a = (a_1^2 + a_2^2)^{1/2}$ the wave number, by $\langle \cdot \rangle$ and $\| \cdot \|$, respectively, the integral and the L^2 -norm on Ω . Finally, taking into account that the stability of m_0 makes sense only in a class of solutions of (2.5) – (2.8) in which m_0 is unique, we exclude any other rigid solution on requiring that

$$\langle u \rangle = \langle v \rangle = 0.$$

3 - A priori estimates

In this section we shall prove an a priori estimate of the perturbation $\theta(P, t)$ to the temperature field. The following result holds [6, 13, 25].

Property 1 - *Let us define the sets*

$$(3.1) \quad \Omega_1 = \{P \in \Omega : \theta(P, t) > T_1 + 1 - \bar{T}(z)\}$$

and

$$(3.2) \quad \Omega_2 = \Omega - \Omega_1 = \{P \in \Omega : \theta(P, t) \leq T_1 + 1 - \bar{T}(z)\}.$$

If

$$(3.3) \quad \theta_0(P) \in W^{2,2}(\Omega)$$

then:

$$(3.4) \quad \theta(P, t) + \bar{T}(z) - T_1 - 1 \leq \bar{\theta}_0, \quad \text{a.e. on } \Omega_1,$$

where $\bar{\theta}_0 = \operatorname{ess\,sup}_{\Omega} [(\theta_0(P) + \bar{T}(z) - T_1 - 1)_+] (< \infty)$.

Proof. The result is proved more naturally on the equation for T , i.e. $(2.3)_3$. To this end, let us consider the following initial boundary value problem (I.B.V.P.):

$$(3.5) \quad \begin{cases} A T_t + \mathbf{v} \cdot \nabla T = \Delta T \\ \nabla \cdot \mathbf{v} = 0 \\ T(P, 0) = T_0(P) \\ T = T_1 + 1 \quad \text{on } z = 0; \quad T = T_1 \quad \text{on } z = 1, \end{cases}$$

where $\mathbf{v}$ is the seepage velocity. Let us consider now the truncated function

$$(3.6) \quad (T(P, t) - T_1 - 1)_+ = \begin{cases} T(P, t) - T_1 - 1 & T(P, t) > T_1 + 1 \\ 0 & T(P, t) \leq T_1 + 1 \end{cases}$$

and let us multiply equation $(3.5)_1$ by

$$[(T(P, t) - T_1 - 1)_+]^{2n-1} \quad (n \in \mathbb{N}).$$

Then, on taking into account $(3.5)_2$ and $(3.5)_4$, since $(T(P, t) - T_1 - 1)_+ \in W^{2,2}(\Omega)$ for all $t \geq 0$, the integration on the periodicity cell Ω and the divergence theorem lead to:

$$(3.7) \quad \begin{aligned} \frac{A}{2n} \frac{d}{dt} \int_{\Omega} [(T(P, t) - T_1 - 1)_+]^{2n} d\Omega = \\ = -(2n-1) \int_{\Omega} [(T(P, t) - T_1 - 1)_+]^{2n-2} |\nabla[(T(P, t) - T_1 - 1)_+]|^2 d\Omega \end{aligned}$$

and hence

$$\frac{d}{dt} \int_{\Omega} [(T(P, t) - T_1 - 1)_+]^{2n} d\Omega \leq 0$$

i.e.

$$(3.8) \quad \left(\int_{\Omega} [(T(P, t) - T_1 - 1)_+]^{2n} d\Omega \right)^{\frac{1}{2n}} \leq \left(\int_{\Omega} [(T_0(P) - T_1 - 1)_+]^{2n} d\Omega \right)^{\frac{1}{2n}}.$$

Now, from (3.8), on setting $n \rightarrow \infty$ [1], one has:

$$(3.9) \quad \operatorname{ess\,sup}_{\Omega} [(T(P, t) - T_1 - 1)_+] \leq \operatorname{ess\,sup}_{\Omega} [(T_0(P) - T_1 - 1)_+].$$

On applying (3.9) to the function $T(P, t) = \theta(P, t) + \bar{T}(z)$, that is a solution of (3.5) with $T_0(P) = \bar{T}(z) + \theta_0(P)$, we obtain:

$$\operatorname{ess\,sup}_{\Omega} [(\theta(P, t) + \bar{T}(z) - T_1 - 1)_+] \leq \operatorname{ess\,sup}_{\Omega} [(\theta_0(P) + \bar{T}(z) - T_1 - 1)_+]$$

from which, on accounting for the definition of the set Ω_1 , one recovers the inequality (3.4).

4 - Linear stability

In this section we shall study the linear stability of m_0 , through the Liapunov direct method ("energy method") [8, 24]. To this end we consider the linear version of the system (2.5), i.e.:

$$(4.1) \quad \begin{cases} \nabla \pi = -R \exp[\Gamma(z - \xi)] \mathbf{u} + R^2 \theta \mathbf{k} \\ \nabla \cdot \mathbf{u} = 0 \\ A \theta_t = \Delta \theta + R w, \end{cases}$$

and introduce the Liapunov function

$$(4.2) \quad V(t) = \frac{A}{2} \|\theta\|^2.$$

On taking into account the boundary conditions (2.7), the time derivative of (4.2) along the solution of the system (4.1) is given by:

$$(4.3) \quad \frac{dV}{dt} = RI - D$$

where

$$(4.4) \quad I = 2 \langle w \theta \rangle, \quad D = \langle \exp[\Gamma(z - \xi)] |\mathbf{u}|^2 \rangle + \|\nabla \theta\|^2.$$

Now, by following the standard energy method, we set

$$(4.5) \quad \frac{1}{R_L} = \max_{\mathcal{H}} \frac{I}{D},$$

where

$$\mathcal{H} = \{\mathbf{u}, \theta : \nabla \cdot \mathbf{u} = 0; \mathbf{u}, \theta \text{ are regular in } \Omega, \text{ periodic in } x \text{ and } y, \text{ of period } 2\pi/a_1, 2\pi/a_2, \text{ satisfying (2.7) and such that } D < \infty\}$$

is the class of *admissible kinematic perturbations*. The maximum (4.5) exists by virtue of the Rionero's theorem [20].

From (4.3), by virtue of (4.5), one immediately obtains

$$(4.6) \quad \frac{dV}{dt} \leq \left(\frac{R - R_L}{R_L} \right) D.$$

The Poincaré inequality [5] leads to the following result of linear stability.

Theorem 1 - *The condition*

$$(4.7) \quad R < R_L$$

ensures the linear, asymptotic and exponential stability of m_0 with respect to the V -norm according to

$$(4.8) \quad V(t) \leq V(0) \exp \left[\left(\frac{R - R_L}{R_L} \right) t \right], \quad t \geq 0.$$

Remark 1 - *Let us observe that the inspection of $(4.1)_1$ leads to*

$$(4.9) \quad \|\mathbf{u}\|^2 \leq c \|\theta\|^2$$

where c is a positive constant. In such a way, by virtue of (4.8) and (4.9), on setting

$$(4.10) \quad V_1(t) = V(t) + \|\mathbf{u}\|^2,$$

it follows that the condition (4.7) guarantees also the linear stability of m_0 with respect to the V_1 -norm.

Remark 2 - The linear operator L of the problem at hand is symmetric with respect to the L^2 -scalar product. This symmetry guarantees that

$$(4.11) \quad R \leq R_L$$

is a necessary and sufficient condition for the linear stability [5].

5 - Numerical results

The Euler-Lagrange equations that solve the variational problem (4.5) are:

$$(5.1) \quad \begin{cases} \Gamma \exp[\Gamma(z - \xi)] w_z + \exp[\Gamma(z - \xi)] \Delta w = R_L \Delta_1 \theta \\ \Delta \theta = -R_L w, \end{cases}$$

to which we append the boundary conditions (2.7)³

Let us notice that, for fluids with temperature dependent viscosity, from the definition of R , i.e.

$$R = \sqrt{\frac{\alpha g \Delta \tau d k \rho_0}{k_1 \mu_0}}$$

it arises that is important to establish a suitable choice for $\mu_0 = \mu(T_0)$, T_0 being a reference temperature. For instance in [17], on considering linear temperature dependent viscosity, the authors used an average viscosity over the layer. But, in [23] it has been pointed out that, for exponential fluids, a suitable choice for the viscosity is $\bar{\mu}$, where $\bar{\mu}$ is the viscosity at the mean of the boundary temperatures. Hence we adopt this last selection. To this end, on choosing $\xi = 1/2$, i.e. $T_0 = (2T_1 + 1)/2$, we solve numerically the equations (5.1) - by using the compound matrix method - and the results obtained in terms of

$$R_{CL}^2 = \min_{a^2} R_L^2$$

³The eigenvalue problem (5.1) is exactly the same of which one arising from the linear instability theory [4].

are listed in table 1.

Table 1 - Critical Rayleigh numbers against Γ , with $\xi = 0.5$

Γ	R_{CL}^2
0	39.4784
0.1	39.4759
0.2	39.4684
0.5	39.4154
1	39.2203
1.5	38.8761
2	38.3574
2.5	37.6347
3	36.6803

6 - Nonlinear stability

In order to study the nonlinear stability of m_0 – through the Liapunov direct method [8] – we choose the Liapunov function (4.2), i.e.

$$(6.1) \quad V = \frac{A}{2} \|\theta\|^2$$

and we evaluate the time derivative of $V(t)$ along the solutions of (2.5), (2.7), (2.8). Then on taking into account the boundary conditions (2.7) and on applying the divergence theorem, the integration on the periodicity cell Ω leads to:

$$(6.2) \quad \frac{dV}{dt} = 2R \langle \theta w \rangle - \langle \exp[\Gamma(z - \xi) - \Gamma\theta] |\mathbf{u}|^2 \rangle - \|\nabla\theta\|^2.$$

Since

$$(6.3) \quad \begin{aligned} - \langle \exp[\Gamma(z - \xi) - \Gamma\theta] |\mathbf{u}|^2 \rangle &= - \int_{\Omega_1} \exp[\Gamma(z - \xi) - \Gamma\theta] |\mathbf{u}|^2 \\ &\quad - \int_{\Omega_2} \exp[\Gamma(z - \xi) - \Gamma\theta] |\mathbf{u}|^2 \end{aligned}$$

on taking into account the inequality (3.4) and the definition (3.2) of the set Ω_2 , from (6.3) we obtain:

$$(6.4) \quad - < \exp[\Gamma(z - \xi) - \Gamma\theta]|\mathbf{u}|^2 > \leq \\ - \exp[-\Gamma(\bar{\theta}_0 + 1)] \int_{\Omega_1} \exp[\Gamma(z - \xi)]|\mathbf{u}|^2 - \exp(-\Gamma) \int_{\Omega_2} \exp[\Gamma(z - \xi)]|\mathbf{u}|^2$$

i.e.

$$(6.5) \quad - < \exp[\Gamma(z - \xi) - \Gamma\theta]|\mathbf{u}|^2 > \leq - \exp[-\Gamma(\bar{\theta}_0 + 1)] < \exp[\Gamma(z - \xi)]|\mathbf{u}|^2 > .$$

Hence, by virtue of (6.5), from (6.2) one has:

$$(6.6) \quad \frac{dV}{dt} \leq 2R < \theta w > - \exp[-\Gamma(\bar{\theta}_0 + 1)] < \exp[\Gamma(z - \xi)]|\mathbf{u}|^2 > - \|\nabla\theta\|^2 .$$

Now, by following the standard energy method, we set

$$(6.7) \quad \frac{1}{R_L} = \max_{\mathcal{H}} \frac{I_1}{D_1},$$

where $\mathbf{u}_1 = \exp[-\Gamma(\bar{\theta}_0 + 1)/2]\mathbf{u}$, $I_1 = 2 < \theta w_1 >$, $D_1 = < \exp[\Gamma(z - \xi)]|\mathbf{u}_1|^2 > + \|\nabla\theta\|^2$ and $\mathcal{H}$ is the class of the *admissible kinematic perturbations* introduced in Section 4. From (6.6), by virtue of (6.7), we obtain

$$(6.8) \quad \frac{dV}{dt} \leq \left(\frac{R \exp[\Gamma(\bar{\theta}_0 + 1)/2] - R_L}{R_L} \right) D_1 .$$

The following nonlinear stability result holds true

Theorem 2 - *In the class of perturbations $\theta(P, t)$ such that*

$$(6.9) \quad \bar{\theta}_0(P) = \operatorname{ess\,sup}_{\Omega} [(\theta_0(P) + \bar{T}(z) - T_1 - 1)_+] < M$$

where M is a positive constant, the condition

$$(6.10) \quad R < R_L \exp[-\Gamma(M+1)/2]$$

guarantees the asymptotic, exponential, nonlinear stability of the conduction solution m_0 with respect to the V -norm, according to the following inequality

$$(6.11) \quad V(t) \leq V(0) \exp \left[\left(\frac{R \exp[\Gamma(M+1)/2] - R_L}{R_L} \right) t \right], \quad t \geq 0.$$

Remark 3 - Let us observe that, by virtue of (3.4), it follows that m_0 is also nonlinearly, asymptotically and exponentially stable with respect to the V_1 -norm given by (4.10).

Remark 4 - The critical value

$$(6.12) \quad R_E = R_L \exp[-\Gamma(M+1)/2]$$

of nonlinear stability defined in (6.10), is less than R_L . However, let us underline that, it is not request that the initial data become vanishingly small, when $R \rightarrow R_E$.

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A NOTE ON WEIGHTED L^2 -NORMS

Nota di Salvatore Rionero¹

Presentata dal Socio Prof. S. Rionero
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Abstract Let $l(>0)$, $\varphi(\geq 0)$ and $g(\geq 0)$ denote a constant and two non-increasing functions, with $0 < g(0) \leq l$. The weighted L^2 - norms $\{A(\varphi) = \int_0^l \varphi(x) f^2 dx, B(\varphi) = \int_0^l g(x) \varphi(x) f^2 dx\}$ are considered. It is shown that $A(1) > 0$ implies $A(\varphi)/B(\varphi) \leq \alpha = A(1)/B(1)$, $\forall \varphi$. When f depends upon another variable t and $\{g(l) = 0, \lim_{t \rightarrow 0} f(x, t) = 0 \forall x \in [0, l]\}$, conditions guaranteeing that $\lim_{t \rightarrow 0} \alpha(t) < \infty$ are obtained and it is shown that, $A^*(\varphi)/B^*(\varphi) \leq A^*(1)/B^*(1)$ where $A^*(\varphi) = \int_0^t A(\varphi, \tau) d\tau$, $B^*(\varphi) = \int_0^t B(\varphi, \tau) d\tau$. Analogous inequalities allowing g and φ to be both increasing or one decreasing and one increasing, are obtained.

Riassunto. Indicate con $l(> 0)$, $\varphi(\geq 0)$ e $g(\geq 0)$ una costante e due funzioni non crescenti con $0 < g(0) \leq l$, si considerano le norme pesate $\{A(\varphi) = \int_0^l \varphi(x) f^2 dx, B(\varphi) = \int_0^l g(x) \varphi(x) f^2 dx\}$. Si mostra che $A(1) > 0$ implica $A(\varphi)/B(\varphi) \leq \alpha = A(1)/B(1)$, $\forall \varphi$. Quando f dipende da un'altra variabile t , si trovano condizioni affinché $\lim_{t \rightarrow 0} \alpha(t) < \infty$ quando $\{g(l) = 0, \lim_{t \rightarrow 0} f(x, t) = 0 \forall x \in [0, l]\}$. Si prova infine che $A^*(\varphi)/B^*(\varphi) \leq A^*(1)/B^*(1)$ ove $A^*(\varphi) = \int_0^t A(\varphi, \tau) d\tau$, $B^*(\varphi) = \int_0^t B(\varphi, \tau) d\tau$. Diseguaglianze analoghe si ottengono considerando g e φ entrambe funzioni crescenti oppure una decrescente e l'altra crescente.

1 - Introduction

As it is well known, weighted norms have been and are extensively used in

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many mathematical fields and, in particular, in the theory of P.D.E.s. So many authors have used and studied weighted norms that it is almost impossible to know all the inequalities involving them [1, 2, 3, 4, 5, 6]. The inequalities between the weighted L^2 -norms $\int_0^l \varphi(x) f^2(x) dx$, of a function f , are rarely considered and, as far as we know, the inequality embodied in (I) has not been introduced before.

The present Note consider the case $\{0 < l \leq \infty; \varphi(x) \geq 0$ non-increasing function; $g \geq 0$ non-increasing function such that $0 < g(0) \leq l\}$. It is shown (Sect. 2) that there exists a positive α - independent of φ - such that

$$(I) \quad \int_0^l \varphi(x) f^2(x) dx \leq \alpha \int_0^l g(x) \varphi(x) f^2(x) dx, \quad \forall \varphi.$$

Successively (Section 3) this inequality is generalized to the case in which f depends also upon another variable (for instance the time) t . In this case α depends upon time t and is not defined when $f(x, 0) = 0 \forall x \in [0, l]$. Conditions guaranteeing that α remains bounded when $g(l) = 0$ and $t \rightarrow 0$ are obtained and it is shown that exists a function $\beta : \tau \in]0, t] \rightarrow \beta(t)$, independent of φ , such that

$$(II) \quad \int_0^t d\tau \int_0^l \varphi(x) f^2(x, \tau) dx \leq \beta(t) \int_0^t d\tau \int_0^l g(x) \varphi(x) f^2(x, \tau) dx.$$

Moreover the case in which l also depends on t is considered. Finally in Section 4 inequalities analogous to (I) are obtained allowing g and φ to be both increasing or one increasing and one decreasing.

2 - Proof of the inequality (I)

Theorem 1 - Let $l < \infty$ denote a positive constant and let

$$(1) \quad \begin{cases} g : x \in [0, l] \rightarrow g(x) \in \mathbb{R}^+, g \in L^1[0, l]; g(x_2) \leq g(x_1), x_2 \geq x_1; 0 < g(0) \leq l \\ \varphi : x \in [0, l] \rightarrow \varphi(x) \in \mathbb{R}^+, \varphi \in L^1[0, l]; \varphi(x_2) \leq \varphi(x_1), x_2 \geq x_1 \\ f : x \in [0, l] \rightarrow f(x) \in \mathbb{R}, f \in L^2[0, l], \int_0^l f^2(x) dx > 0. \end{cases}$$

Then the inequality (I) holds where α , independent of φ , is given by

$$(2) \quad \alpha = \frac{\int_0^l f^2(x) dx}{\int_0^l g(x) f^2(x) dx}.$$

Proof. From (2) it turns out that

$$(3) \quad a \int_0^l f^2(x) dx = \int_0^l g(x) f^2(x) dx,$$

where $a = \frac{1}{\alpha}$ and therefore

$$(4) \quad \int_0^l [g(x) - a] f^2(x) dx = 0.$$

But (3) implies $0 \leq g(l) < a < g(0) \leq l$, therefore from (4) it follows that

$$(5) \quad \int_0^{x_0} [g(x) - a] f^2(x) dx = \int_{x_0}^l [a - g(x)] f^2(x) dx,$$

where x_0 is such that $g(x_0) = a$.

Now, because φ is a non-increasing function and

$$\begin{cases} \varphi \geq 0 & \forall x \in [0, l] \\ g(x) \geq a & \forall x \in [0, x_0] \\ g(x) \leq a & \forall x \in [x_0, l], \end{cases}$$

we obtain

$$(6) \quad \begin{cases} \int_0^{x_0} [g(x) - a] \varphi(x) f^2(x) dx \geq \varphi(x_0) \int_0^{x_0} [g(x) - a] f^2(x) dx \\ \int_{x_0}^l [a - g(x)] \varphi(x) f^2(x) dx \leq \varphi(x_0) \int_{x_0}^l [a - g(x)] f^2(x) dx. \end{cases}$$

On multiplying by $\varphi(x_0)$ both sides of (5) and taking into account (6) it turns out that

$$\int_0^{x_0} [g(x) - a]\varphi(x)f^2(x)dx \geq \int_{x_0}^l [a - g(x)]\varphi(x)f^2(x)dx$$

i.e.

$$\int_0^l [g(x) - a]\varphi(x)f^2(x)dx \geq 0$$

and (I) is completely proved.

Remark 1 - We notice that

i) (I) holds for any α when $f = 0$ a.e. on $[0, l]$;

ii) α depends on f and is bounded when $g(l) \neq 0$. In fact $g(l) \neq 0 \Rightarrow \frac{1}{g(0)} \leq \alpha \leq$

$\frac{1}{g(l)}$ while $g(l) = 0$ implies only $\alpha \geq \frac{1}{g(0)}$;

iii) $\varphi \geq 0 \forall x \in [0, l]$ implies

$$(7) \quad l \int_0^l \varphi(x)f^2(x)dx \geq \int_0^l g(x)\varphi(x)f^2(x)dx, \quad \forall \varphi \geq 0.$$

iv) In the sequel we will consider only the case $l < \infty$. But we notice that theorem 1 continues to hold also for $l = \infty$. In this case, of course, $g(0)$ has only to be positive.

3 - Proof of the inequality (II)

Let us consider now the case in which f depends also upon an another variable (for instance the time) t . This case is of interest in many applications. In particular, when f depends upon the time t , and so does l , the inequality (I) with $\{\varphi = [l(t) - x]^s, s > 0\}$ is of interest in free-boundary value problems [7, 8, 9].

Theorem 2 - Let $l < \infty, T < \infty$ denote two positive constants; let $(1)_1 - (1)_2$ hold and

$$(8) \quad \begin{cases} D(t) = [0, l] \times [0, t], & 0 < t \leq T; \\ f : (x, t) \in D \rightarrow f(x, t) \in \mathbb{R}, f \in L^2[D(T)], \int_{D(t)} f^2(x, \tau) dD > 0. \end{cases}$$

Then the inequality (II) holds where β , independent of s , is given by

$$(9) \quad \beta : t > 0 \rightarrow \beta(t) = \frac{\int_{D(t)} f^2(x, \tau) dD}{\int_{D(t)} g(x) f^2(x, \tau) dD}.$$

Proof. On setting $b = \frac{1}{\beta}$, it turns out that $b < g(0)$ and

$$(10) \quad \int_0^t d\tau \int_0^{x_0} [g(x) - b] f^2(x, \tau) dx = \int_0^t d\tau \int_{x_0}^l [b - g(x)] f^2(x, \tau) dx,$$

where x_0 is such that $g(x_0) = b$. Taking into account that φ is a non-increasing function, the proof is immediately obtained by following step by step the procedure used in proving (I).

Remark 2 - Let f be sufficiently smooth. Then

- i) $\int_{D(t)} f^2(x, \tau) dD = 0 \Rightarrow f = 0, \quad \forall (x, \tau) \in D(t);$
- ii) $\int_{D(t)} f^2(x, \tau) dD > 0, \quad \forall t \in]0, T] \Rightarrow \beta \in C^1]0, T];$
- iii) $\int_{D(t)} f^2(x, \tau) dD > 0, \quad \forall t \in]0, T]$ and $\lim_{t \rightarrow t_0} \beta(t) = \beta_0 < \infty$ imply $\beta \in L^2[0, T];$
- iv) $\lim_{t \rightarrow 0} \beta(t) = \lim_{t \rightarrow 0} \alpha(t) = \lim_{t \rightarrow 0} \frac{\int_0^l f^2(x, t) dx}{\int_0^l g(x) f^2(x, t) dx};$
- v) $\lim_{t \rightarrow 0} \alpha(t) < \infty \Rightarrow \alpha, \beta \in L^2[0, T],$ also when $g(l) = 0;$
- vi) $f(x, 0) = \eta = \text{const} \neq 0 \Rightarrow \lim_{t \rightarrow 0} \alpha(t) = l / \int_0^l g(x) dx.$

Theorem 3 - Let $\mathbb{R}^+$ be $[0, \infty[$ and let I be the subset of $L^2_{loc}[(\mathbb{R}^+)^2]$ constituted by the functions

$$f : (x, t) \in \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow f(x, t) \in \mathbb{R}$$

smooth and such that

$$(11) \quad \begin{cases} \int_{D(t)} f^2(x, \tau) dD > 0, & t > 0 \\ f(x, 0) = 0, & \forall x \in \mathbb{R}^+ \\ \lim_{t \rightarrow 0} \beta(t) < \infty. \end{cases}$$

Then $f \in L^2_{loc}[(\mathbb{R}^+)^2]$ and

$$(12) \quad \begin{cases} t^p g(t) \psi_1(x, t) \leq |f(x, t)| \leq g(t) \psi_2(x, t), & x \in \mathbb{R}, t \in [0, t_0], p < \frac{1}{2} \\ 0 \leq g \in L^2[0, t_0], 0 \leq \psi_2 \in L^2_{loc}\{\mathbb{R}^+ \times [0, t_0]\}, \int_0^i \psi_i^2(x, 0) dx > 0, & i = 1, 2; \end{cases}$$

t_0 being a positive constant, imply $\{f \in I; \alpha, \beta \in L^1_{loc}(\mathbb{R}^+)\}$ also when $g(t) = 0$.

Proof - Because of iv) of Remark 2, the proof is straightforward.

Remark 3 - Let us remark that the assumptions of theorem 3 allow f a very general behaviour. In fact

- i) $f \in L^2_{loc}(\mathbb{R}^+)^2$ allows f to grow exponentially when $x \rightarrow \infty$ or $t \rightarrow \infty$;
- ii) assumptions (12) allow f to tend to zero like At^q , A and q being positive constants, when $t \rightarrow 0$;
- iii) $f = c = \text{const} \neq 0, \forall t \in (\mathbb{R}^+)^2 \Rightarrow f \notin I$.

We end by noticing that

$$\begin{cases} g(t) \psi(x) \leq |f| \leq mg(t) \psi(x) & (x, t) \in (\mathbb{R}^+)^2 \\ g(t) = \exp[\exp(t) - \exp(1/t)], & t \in \mathbb{R}^+ \\ \psi(x) = \exp[\exp(\exp(x))] & x \in \mathbb{R}^+ \\ m = \text{positive constant} \geq 1 \end{cases}$$

imply (12).

Remark 4 - Let us notice that

i) if

$$(13) \quad \begin{cases} l : t \in [0, T] \rightarrow l(t) \in \mathbb{R}^+, T = \text{const.} > 0, l \in C[0, T] \\ f : x \in [0, l(t)], t \in [0, T] \rightarrow f(x, t) \in \mathbb{R}, \int_0^{l(t)} f^2(x, t) dx > 0, \forall t \in [0, T] \end{cases}$$

then (I) continues to hold at any instant $t \in [0, T]$ with α depending on t according to

$$(14) \quad \alpha(t) = \frac{\int_0^{l(t)} f^2(x, t) dx}{\int_0^{l(t)} g(x) f^2(x, t) dx}, \quad \forall t \in [0, T];$$

- ii) if $\int_0^{l(t)} f^2(x, t) dx > 0, t \in]0, T]; l(0) > 0$ and $f^2(x, 0) \equiv 0, x \in [0, l(0)]$; then α , although is not defined for $t = 0$, belongs to $L^2[0, T]$ if (12) holds;
- iii) inequalities analogous to (I) and (II) can be obtained also on domains of $\mathbb{R}^3$.

4 - Inequalities for $\int_0^l \varphi f^2 dx$ when g and φ are both increasing or one increasing and one decreasing

Following step by step the proof of theorem 1, the following theorems can be easily obtained.

Theorem 4 - Let $l(< \infty)$ be a positive constant and $(1)_4$ hold. If g and φ are such that

$$(15) \quad \begin{cases} g : x \in [0, l] \rightarrow g(x) \in \mathbb{R}^+, g \in L^1[0, l]; x_1 \leq x_2, g(x_1) \leq g(x_2); 0 < g(0) < g(l) \leq l \\ \varphi : x \in [0, l] \rightarrow \varphi(x) \in \mathbb{R}^+, \varphi \in L^1[0, l]; x_1 \leq x_2, \varphi(x_1) \leq \varphi(x_2). \end{cases}$$

Then (I), with α given by (2), continues to hold.

Theorem 5 - Let $l(< \infty)$ be a positive constant and $(1)_1, (1)_3$ and $(15)_2$ hold. Then

$$(16) \quad \int_0^l \varphi(x) f^2(x) dx \geq \alpha \int_0^l g(x) \varphi(x) f^2(x),$$

with α given by (2).

Theorem 6 - Let $l(< \infty)$ be a positive constant and $(1)_3, (1)_3$ and $(15)_1$ hold. Then (16) hold with α given by (2).

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An outline of the geology of External Dinarides and their Mesozoic - Early Tertiary facies

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Presentata dal socio Bruno D'Argenio
(Adunanza del 7 nov. 1998)

Key words: Dinarides, Regional Geology, Stratigraphy, Mesozoic, Tertiary.

Abstract - Brief introduction to the geology of the Southern Dinarides, with special regard to Crna Gora (Montenegro) and adjacent parts of Serbia and Croatia which form an important sector of this orogenic belt facing Southern Apennines and their Apulian foreland, where up to dated geologic information is scarce and not always reliable. An appendix illustrates the regional geology and in particular the stratigraphy and facies analysis, with much geologic and geomorphologic information (up to now unpublished) on the region along a 1000 km field trip commencing from and ending in Dubrovnik.

Riassunto - Breve introduzione alla geologia delle Dinaridi meridionali, con particolare riguardo al Montenegro (Crna Gora), un importante settore orogenico situato di fronte all'Appennino meridionale e al suo avampaese, dove le conoscenze geologiche moderne sono scarse e non sempre attendibili. Un'appendice illustra la geologia regionale del Montenegro e di zone adiacenti della Serbia e della Croazia, con particolare riguardo alla stratigrafia e all'analisi delle facies, mediante un'escursione di 1000 Km lungo la quale sono presentate numerose informazioni geologiche e geomorfologiche finora inedite, che inizia e termina a Dubrovnik.

1. - INTRODUCTION

This is a brief introduction to the geology of Southern Dinarides of Montenegro (Crna Gora) and adjacent sectors of Serbia and Croatia. Here several large tectonic units have been individuated forming both the internal and the external part of this orogenic belt. Not always these units may be considered also as facies zones, being the criteria used in their identification not homogeneous.

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On the base of our present knowledge, only from the Middle Triassic we can distinguish throughout the Dinarides a number of sedimentary domains which, from the geometry of the sedimentary bodies, appear as a system of shallow water carbonate platforms and intervening basinal areas, variably interconnecting and branching. Their arrangement changes along the strike of the belt, and these characteristics makes correlation with the Hellenides and the Apennines quite difficult. In Fig. 2 a tentative correlation of Dinarides external domains with those of Hellenides is given.

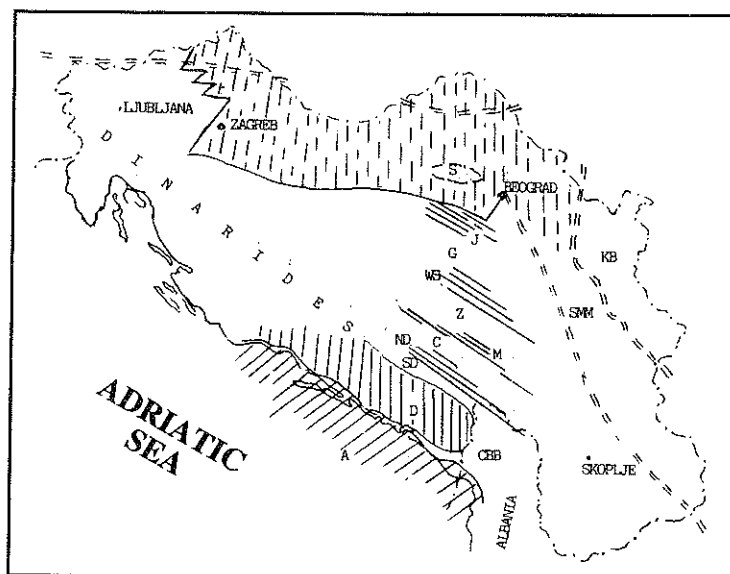


Figure 1. Main facies domains along a transect in Montenegro-Serbia Dinarides (from which several paleogeographic units may be restored). S) Srem Basin; J) Jadar Carbonate Platform; G) Gucevo Basin; WS) West Serbia C.P.; Z) Zlatac Basin; M) Mihajlovici Carbonate Platform; C) Ceotina Basin; ND) North Durmitor C.P.; SD) South Durmitor Basin; D) Dinaric Carbonate Platform; CBB) Cukali-Budva Basin; A) Adriatic C.P. KB) Karpato - Balkanides; SMM) Serbo - Macedonian Massif.

2. - THE DINARIC OROGENIC BELT

2.1 - Dinaric Domains

Since the Early Jurassic the widening of the oceanic Tethys and its wedging westwards has given rise to a differentiation of the internal domains (central Tethys realm) from the external ones (southern continental margin of Tethys) where the previous Triassic paleogeographic pattern remains more or less the same.

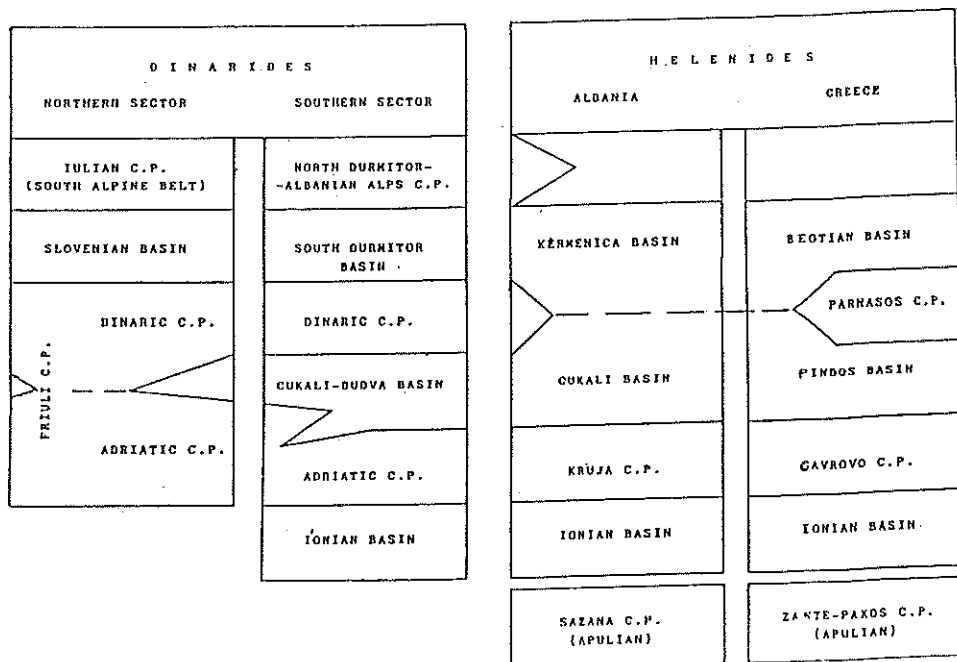


Figure 2. Tentative correlation of the Mesozoic Dinaric and Hellenic facies domains (from RADOICIC, 1987, modified after CATI et al., 1987, for Friuli area).

The Serbia-Montenegro sector is a good example of the sedimentary evolution of the Mesozoic Dinarides. In this sector, from the North, the following Triassic domains may be distinguished (Figs. 1, 2):

-a. Internal domains:

- South Backa (Carbonate Platform, known only in subsurface)
- Srem (Basin)
- Jadar (Carbonate Platform).
- Gucevo (Basin)
- West Serbia (Carbonate Platform)
- Lim basin (sensu lato), including:
 - Zlata (Basin)
 - Sandzak (Carbonate Platform)
 - Ceotina (Basin)

b. External domains

- Middle Dinaric Belt, including:
 - North Durmitor (Carbonate Platform)
 - South Durmitor (Basin)
- External Dinaric Belt, including:
 - Dinaric (Carbonate Platform).

- Cukali-Budva (Basin)
- Adriatic (Carbonate Platform)
- Ionian (Basin)

During the Triassic the complex Lim Basin (RAMPNOUX, 1974), formed by basinal areas with intervening carbonate platform, acted as an important seaway (Southwestern Serbia-Northern Montenegro). After the Early Jurassic events the Lim Basin may be considered as facing the Tethyan oceanic domains (RADOICIC, 1982).

2.2. Mesozoic - Early Tertiary domains of Southern External Dinarides.

The following is a brief description of the main sedimentary domains that may be individuated in the Southern Sector of External Dinarides. The pertaining rocks form at present tectonic units thrust toward the Adriatic foreland; their original boundaries very often correspond to present tectonic lines and frequently are obscured by thrusting.

2.2.1. North Durmitor-Albanian Alps Carbonate Platform

The North Durmitor-Albanian Alps is a Carbonate Platform Domain spread in the transition area between Internal and External Dinarides. This platform was of shorter duration, as compared to the more external carbonate platforms but lasted more as compared to the platforms of the internal domain that did not survive Liassic times (fig.3).

The North Durmitor Platform faded during uppermost Malm-earliest Neocomian tectonic events which caused platform drowning. Early carbonate platform stages are characterized by thick carbonate series, while outcropping Jurassic sequences are of variable thickness, from one to several hundred meters. Rare and not significant bauxite deposits (suggesting long lasting emersions) occur between Middle and Upper Triassic, and even between Upper Liassic (Ammonitico Rosso) and Upper Malm (shallow water limestones).

2.2.2. South Durmitor Basin

The South Durmitor Basin is among the significant features in the Dinaridic belt, and may be considered, together with the Carbonate Platform of North Durmitor-Albanian Alps, as an Internides/Externides transitional zone. This is a large and complex unit with a very irregular geometry, especially in the Cretaceous and Early Tertiary times. It is also strongly deformed.

The Pre-Upper Cretaceous sequences of South Durmitor units are not sufficiently known. Toward southeast, this basin seems to merge in the Cukali-Budva Basin.

2.2.3. Dinaric Carbonate Platform

The Dinaric Carbonate Platform shows sedimentary characters comparable with the Campania- Lucania Carbonate Platform of the Southern Apennines. It

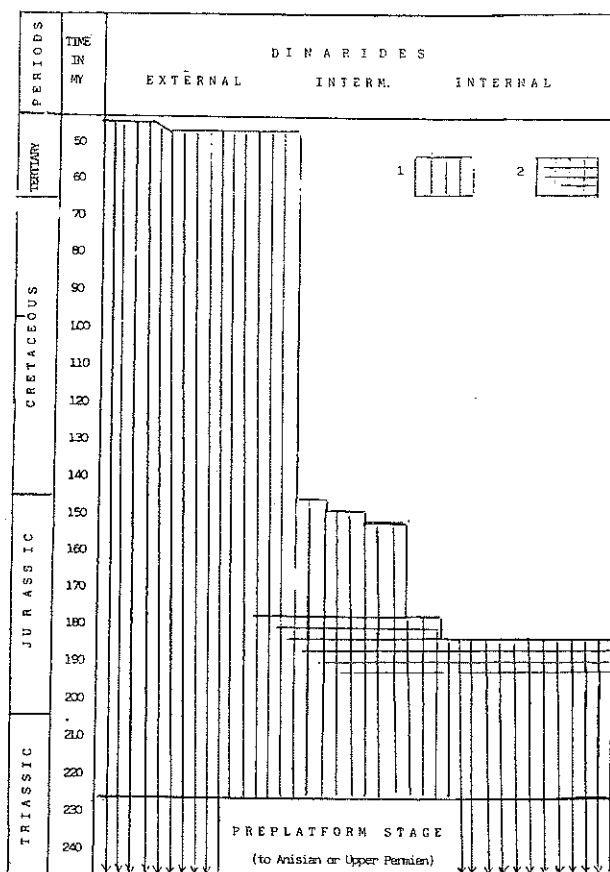


Figure 3. Time of Carbonate Platform development in the Dinarides of Montenegro-West Serbia (1) and their drowning and local recovering, as suggested by the spreading of the Ammonitico Rosso (2). Basins are not indicated (from RADOICIC, 1987).

forms a very large domain that extends from the Montenegro/Albania boundary to the northwest, broadening in Slovenia and developing up to the Friuli. This means that this large unit reaches more than 600 km in length and has a present width of 150 to 250 km. The Dinaric Carbonate Platform lasted more than 180 MY, from Middle Triassic (excluding pre-platform stage) into Middle Eocene. The thickness of carbonate platform body exceeds several thousand meters. In such a large unit minor basin type areas can be recognized (Upper Jurassic Lemes and Upper Cretaceous-Miocene Zeta intraplatform furrows).

Characteristic of the Jurassic platform history, is the development of small satellite basins in its marginal areas (Fig. 4). These small units (Zalomka-Gacko and Lovcen-Rumija and Planinica intramarginal depression in Slovenia) were

of variable life span and different in size. Some reached the Early Dogger, others survived up to Malm, to earliest Neocomian or to Aptian-Albian).

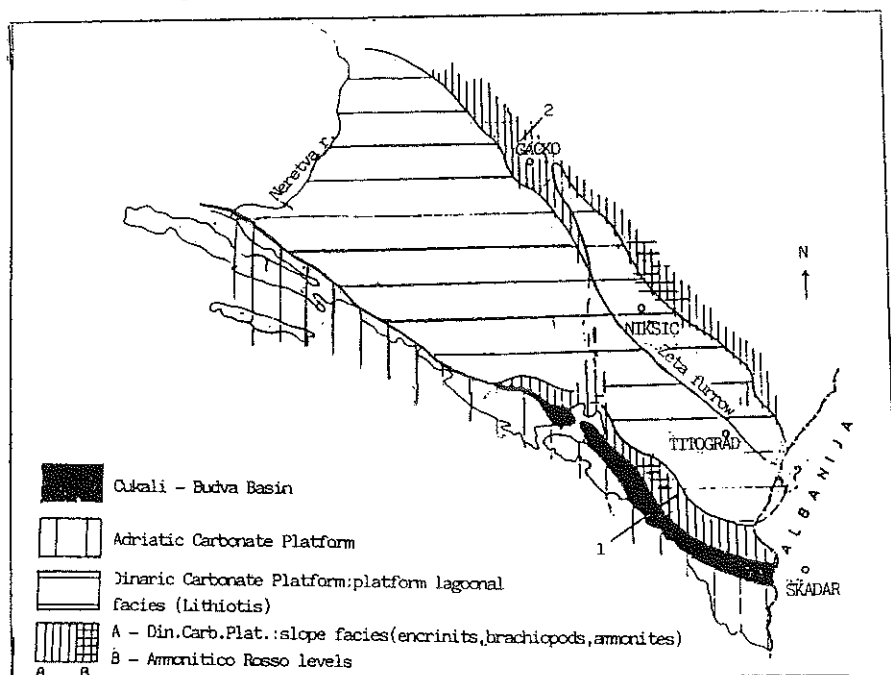


Figure 4. Sketch of the southern sector of the Dinaric Carbonate Platform at Liassic times. 1) Lovcen-Rumija and 2) Zalomka-Gacko adjacent basins (from RADOICIC, 1987).

The Dinaric Carbonate Platform sequences are characterized by a cyclic organization of the strata with repeated emersions, evidenced by stratigraphic gaps during the Mesozoic and Paleogene, with red bauxites occurring in the Triassic, Jurassic, Cretaceous and Paleocene, and white bauxite in the Lower Cretaceous of Southwestern Montenegro (Fig. 5).

The sedimentary sequences of the Dinaric domain consist of Lower Triassic terrigenous sediments and carbonates, Middle Triassic (Anisian) shallow water carbonates, Ammonitico Rosso Facies (Han Bulog), as well as other nodular or bedded limestones. In the SE Dinaric domain also terrigenous sediments occur (so called Anisian Flysch). These deposits pass into the Ladinian Porphyrite-Chert Formation or, at places, into organogenic limestones.

Shallow-water carbonates are widespread during the Late Triassic (with a prevalent cyclothemic organization) and continue up to Late Cretaceous or Middle Eocene. After the Middle Eocene terrigenous sediments developed over the rapidly drowning shallow-water areas (RADOICIC et al., 1991; de CAPOA & RADOICIC, 1994a).

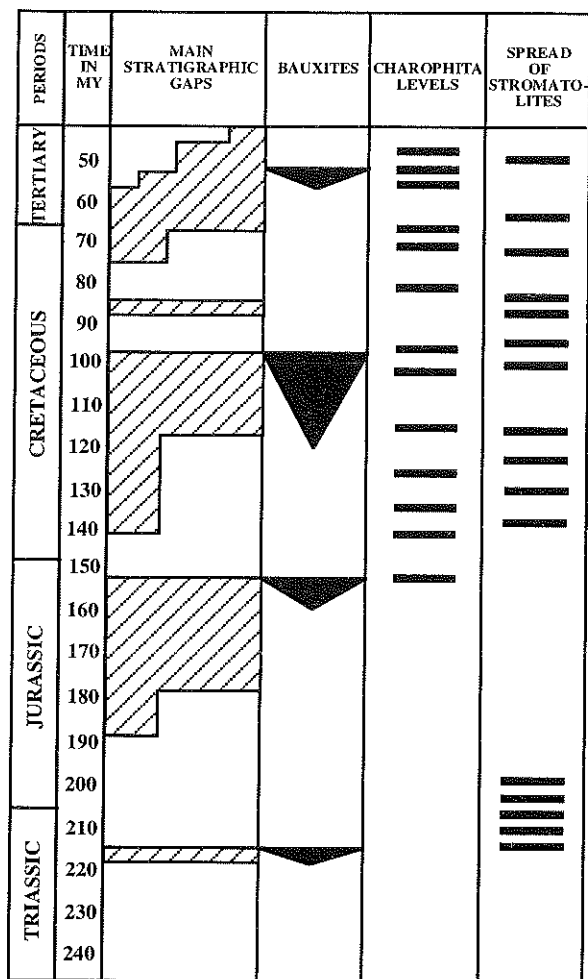


Figure 5. Distribution of the main stratigraphic gaps, bauxites, Charophyta levels and stromatolites-cryptalgal episodes in the Dinaridic Carbonate Platform. Anisian bauxite is not indicated (from RADOICIC, 1982).

2.2.4. Cukali-Budva Basin

The Cukali-Budva Basin is the earliest and most longeval domain in the Dinaridic paleogeography, being an intracontinental basin in Triassic times, with basinal sedimentation from Early Middle Triassic or Late Werfenian, continuing into the Upper Miocene (de CAPOA & RADOICIC, 1994).

The northwesternmost outcrop of the Cukali-Budva Domain is the area between Boka Kotorska and Dubrovnik. From this area, two carbonate Platform Domains, Dinaric and Adriatic, are in tectonic contact (Dinaric Carbonate Platform Domain thrust directly over Cukali-Budva domains, as well as over the Adriatic Carbonate Platform domains).

Undoubtedly the Cukali-Budva Domain becomes narrower and somewhat shallower toward the Northwest: facies analysis suggests a shallowing of the basin along its axis from Budva to Boka Kotorska area. In its prolongation from Neretva toward NW and W it appears as a minor feature (intraplatform Split furrow branch toward NW and Vis furrow branch toward W, Fig.2). Even if it is presumable that Vis furrow was only a basinal tongue, it cannot be excluded that other basinal facies tectonically obscured may exist.

Cukali-Budva Basin sequences consist of Anisian "Flysch", Ladinian Porphyrite-Chert Formation (Type locality in Budva area, CIRIC, 1954), Ladinian to uppermost Cretaceous pelagic cherty limestones and resedimented carbonates and Paleogene to Miocene pelagic silty and marly sediments, while turbiditic sediments develop in some sectors. The above Mesozoic sequences have very variable thickness, from few to several hundred meters.

2.2.5. Adriatic Carbonate Platform

The Adriatic Carbonate Platform is very elongated and extends along the Adriatic coastal belt, through Albania (Kruja) and Greece (Gavrovo) to the East (Dodecanese Islands and South Turkey). A large part of the southern platform sector is submerged in the present Adriatic Sea. Its external margin to the South may be recognized only in the Susac Island.

Along the coastal belt of Croatia and Montenegro Upper Cretaceous outcrops prevail; the oldest known rocks are Middle Triassic Porphyrite-Chert and Carnian evaporitic sequences in the Island of Vis. In the Bar area Ladinian perireef carbonates (Lisanj) and Upper –Triassic cyclothemic limestones (Stari Bar) occurs while younger sequences (Dogger to Paleogene) consist of limestones and of Paleogene to Upper Miocene terrigenous deposits (de CAPOA et al., 1991, 1995). Anhydrites, previously attributed to Lower Cretaceous (CANOVIC, 1969) are most likely Carnian (or Permo-Triassic) in age (RADOICIC, 1987).

2.2.6. Jonian Basin

The Jonian Basin is only known in the Hellenic realm. Its northward continuation into the Dinarides is not documented, except the outcrops of the Palagruza Island (?) and some offshore wells (Juzni Jadran wells, unpublished data).

3. - GEOMORPHOLOGY³

The Montenegro (Crna Gora) and Eastern Hercegovina are characterized by complex geomorphology which results in high variability and in the coexistence of different morphotypes which carry evidence of different morphochronology. Moreover, being the dominant rocks carbonatic in nature they have given to the relief a typical karstic overprint.

3.1 *Adriatic coastal belt (from Dubrovnik to Scutari Lake)*

This sector is characterized by more or less old abrasional morphology, while the Pleistocenian glacial events have left good evidences in the higher mountains facing the lower coastal belt.

The older, well documented abrasion surface occurs at 720 m of elevation (Pastrovicka Gora, Brajici-Rumija). A lower level may be recognized at 400-450 m (Dubrovnik, Trebinje and Budva surroundings, Crmnica).

The lowest abrasion surface is represented in the coastal relief from Cavtat to South Boka (Adriatic Carbonatic belt).

Coastal cliffs alternating to beaches characterize the present abrasion relief. Particularly interesting are the polyphasic cliffs of the Dubrovnik-Budva coast, along which other morphological elements may be observed (abrasion caves, sand bars).

3.2 *Karstic region*

The Crna Gora (Montenegro)-Hercegovina is the type-region for the karstic morphology. In this region true *karstic deserts* are fully developed (see stop 5).

In the Crna Gora-Eastern Hercegovina three Karst types may be distinguished: a) Holokarst (Stara Crna Gora = Old Montenegro-Orjen Mountains, Pusti Lisac and Njegosi Mountains). b) Dolomitic Karst (Western Montenegro and sector South of Titograd, like in Ljesanska Nahija and surroundings of Rijeka Crnojevic) with many *uvalas* and an even relief which contrasts with the very dissected morphology of the holokarst with its "rocky sea" as a dominant in the paisage. c) High Mountain Karst, North of the line Cijevna Canyon-Gacko (Prekornica, Vojnik, Durmitor, Maglic, Volujak mountains).

The Montenegro-Hercegovina includes among the largest karstic poljes, of the Dinarides, some of which will be crossed during the excursion (Njegusko Polje, Cetinjsko Polje, Niksicko Polje, and Plansko Polje).

Noticeable fluviokarstic features are old, largely dry valleys like that of (a) River Cetinje (Cetinjska Rijeka), (b) River Boka (Bokeljska Rijeka) with two

³ Text kindly written by M. Ljesevic, Beograd (from 3.1 to 3.4), and by J. Petrovic, Beograd (3.5).

branches, (c) Surdup, i.e. old valley of Gornja Zeta River and, (d) old valley of Trebisnjica, between Gatacko Polje and Bileca.

Many large caves are known in this region. Duboki Do Hole has been explored down to 400 m of depth. In the Mountain Durmitor moreover the deepest cave of Yugoslavia occurs (897 m of depth). Many long caves are also known, which develop from 2 to 3.5 km (Lipska pecina, Grabocica pecina, and Zacirska pecina), the longest has been explored for 8 km in South Hercegovina (Vjeternica pecina).

3.3 *Fluvial Morphology*

Besides the fluviokarstic features above mentioned, many fluvial denudation forms may be observed in Montenegro, being at present in a stage of formation.

Two main forms dominate: old fluviodenudation high plateaus and canyon valleys carved within these plateaus. Such canyon valleys are the most impressive morphologic elements of the Northwestern Montenegro and North Hercegovina: Sutjeska Canyon (500 m deep), Susica Canyon (600 m), Moraca Canyon (700 m deep), Piva Canyon (up to 1100 m deep) and Tara Canyon (up to 1300 m deep).

3.4 *Glacial morphology*

Large part of the crossed region has been covered by ice during the Pleistocene.

Glacial cirques, glacial valleys and morains are common (Mountain Orje, Mountain Lovcen) with noticeable glacial valleys in the mountains of Moraca, Prokletije, Durmitor and Volujak. The length of such valleys (sometime exceeding 20 km, with about 200 m thick morains) bears evidences of the intensity of Late Pleistocene glaciation. In the Durmitor Lake plain, a 30 km² Late Pleistocene glacier has been suggested. In the lower areas during the Late Pleistocene interglacial and during the Holocene glacial material was redeposited.

Fluvioglacial terraces are well preserved in the valleys of several rivers (Moraca, Tara, Komarnica, Piva and Sutjeska). Such terraces develop at about 120 m above the present river bed.

Water wells in the Piva and Drina valleys have shown up to 90 m of fluvioglacial deposits below the present river bed.

Large part of the River Zeta plain is formed by fluvioglacial conglomerates deposited in large lakes while in the Bjelopavlici plain (NW of Titograd). Fluvioglacial clays and sands form the Zeta plain.

3.5 *Trebisnjica River System*

Among the unique features of the underground karst the Trebisnjica River must be mentioned. This is the longest subterranean river of Europe one of the longest in the World. Its origin is in the Lebrsnik mountain area (South Durmitor Basin) and it reemerges four times before the mouth, located near the town of Dubrovnik. The five segments of this very long and peculiar river are (1) Musnica River, from Lebrsnik into the Gacko Polje (25,5 km); (2) Kljuc River, west of Stepen, short segment; (3) Fatnica River, emerges and disappears within the Fatnica Polje (4,5 km); (4) Trebisnjica River, near Bileca, the longest segment (today an artificial lake), whose outlet forms a group of important springs ; (5) Ombla River, North of Dubrovnik, and a group of large springs in the Adriatic Sea (but part of the waters have different spring groups as independent outlet).

4. - SEDIMENTARY EVOLUTION

4.1. *Paleozoic sequences and facies*

The main pre-Mesozoic, mostly Late Paleozoic, rocks of Dinarides outcrop in a large belt developing along the strike of the chain with a median position (Fig.6). The principal outcrops are known from Macedonia to Slovenia. Several zones have been recognized by different Authors. Namely CHOROWICZ (1977) and RAMPNOUX (1974) have suggested a division in three main domains (Fig. 6). Actually it seems that the reasons for such division are more in the differences of the overlying Mesozoic than in the Paleozoic itself.

4.1.1. Vardar – Pannonic Domain

This domain includes also pre-Cambrian and Early Paleozoic rocks (granites, metamorphites and sedimentary rocks) and is often affected by Paleozoic and Alpine metamorphism. The Late Paleozoic is represented by terrigenous Carboniferous deposits (Culm facies), Lower Permian deposits and Upper Permian carbonates.

4.1.2. Intermediate Domain

In this domain the Late Paleozoic outcrops quite diffusely and the terrigenous Culm deposits (Lower Carboniferous) are widely spread. The Upper Carboniferous is represented by terrigenous sediments and shallow-water very fossiliferous carbonates. The Middle-Upper Permian consists of clastic and organogenic (foraminiferal, algal, brachiopodal) limestones and rests unconformably on its late Paleozoic substrate.

4.1.3. External Domain

Similar late Paleozoic sediments appear also in this domain. It must be noticed also that in Southern Montenegro the same deposits form olistolithes in the "Anisian Flysch".

Red clastic facies of the uppermost Upper Permian, containing evaporites, continues into Lower Triassic. Therefore, it is often referred to as Permo-Triassic or Permo-Werfenian.

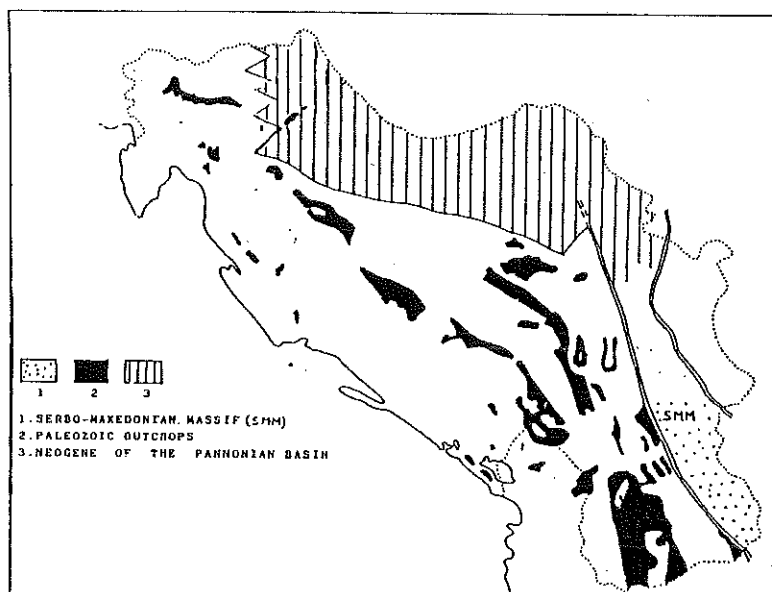


Figure 6. Pre-Mesozoic rocks of Yugoslavia (modified after RAMPNOUX, 1974 and HOROWICZ, 1977).

4.2.-Mesozoic Sedimentary Domains

4.2.1. Lim Basin (Fig. 7)

The Triassic rocks of this domain may be grouped after RAMPNOUX (1974) in three zones from NE to SW: Zlata, Mihajlovici and Cetina (i.e. Zlata Basin, Sandzak or Mihajlovici Carbonate Platform and Cetina Basin).

Lower Triassic – it is given mostly by (a) reddish micaceous sandstones and pelites with *Claraia clarae*, *Homomya fassaensis*, *Rectocornuspira kahleri* (Sesian) (b) very coarse terrigenous deposits (conglomerates and sandstones) and (c) by impure very fossiliferous sandy, marly and dolomitic limestones with *Turbo rectecostatus*, *Naticella costata*, *Meandrospira pusilla*, *Ammodiscus incertus*, *Glomospira* etc. (Campil).

Middle Triassic - In the northeastern and southwestern subzones thick, barren and organogenic limestones (200-300 m) develop, while in the central zone there are thinner (20-30 m) dolomitic limestone deposits.

In the lower part, fossils are obscured by recrystallization. Middle Anisian occurs as very fossiliferous limestones. Besides macrofossils (*Rhynchonella*, mollusks, crinoids) it contains mostly very rich foraminiferal-algal association:

age \ subzone	CEOTINA	MIHAJLOVIĆI	ZLATAR
Liassic		Rosso Ammonitico Brach.Crin.lms.	
Upper Triassic	Well bedded lmp., cherty lms. /with intercal.of resedim.calcaren. and microbreccia/ (Halobia Radiolaria, Conodonts)	Megalodon and Foram.Algal lms. Reef /Coral, Sponge, Hydroz.Stromat./	Well bedded platy lms., cherty lms., and jespes /with some resedim. calcarenites toward central subzones/ (Halobia, Radiolaria, Conodonts)
Ladinian	t u f f s		p o r p h y r i t e s
	Han Bulog	and	hemipelagic lms.
Anisian	Massif /recryst./ organogenous lms.	dolomitic lms.	Massif organog. lms.
Werraenian		Possiliferous sandy and dolomitic lms. micaceous lms., claystone	oid. lms.
Permo-Werf.	red sandstone and	micaceous sandstone	
	North Durm. Carb. Platform		W. Serbia Carb. Platform

Figure 7. Facies distribution in the Lim Basin. Note that the carbonate platform facies in the central part of the basin were followed by drowning in Liassic times as evidenced by encrinurites and Rosso Ammonitico.

Meandrospira dinarica, *Pilammina densa*, *Earlandinita oberhauseri*, *Duostominidae*, *Endothyridae* and other foraminifers; *Macroporella alpina*, *Diploporella hexaster*, *Teutloporella hirsuta*, *Oligoporella prisca* and other algae etc.; conodonts: *Gladigondolella tethydis*, *G. budurovi*, *Paragondolella bifurcata*, *P. hambulogi* and others (RAMPNOUX, 1974; SUDAR, 1986). For detailed lists of foraminifera, algae and conodonts from the Triassic sediments in many localities of Lim Basin, and also other internal domains, see SUDAR (1986).

At the end of Anisian (Illyrian) in all three zones, bedded micrites, platy to nodular and red nodular (Han Bulog) ammonitic and pelagic-hemipelagic limestones follow. The latter mostly contain radiolarians and pelagic pelecypods; in the Han Bulog type limestones besides ammonites (*Paraceratites trinodosus* zone), *Globochaete*, *Ophtalmidium*, *Fronicularia*, *Nodosaria*, *Lenticulina* and many other foraminifers like in the Middle Anisian limestones occur.

The Ladinian is represented by the Porphyrite-Chert Formation, also spread in the lowermost Ladinian over the entire Lim domain, with large amount of Porphyrites and related tuffites. Then only in the basinal units (Zlatar, Ceotina) the Porphyrite-Chert Formation follows. In the bedded and thin-bedded limestones of this Formation radiolarians, sponge spicules, pelagic pelecypods

(*Daonella*, *Posidonia*) are present and, in some intercalated calcarenites or microbreccias also benthonic foraminifers and various algae and metazoan debris may be found. However, in the middle of the Mihajlovici facies zone, Ladinian reefs developed, with *Colospongia catemulata*, *Follicatena cautica*, *Bacinella ordinata*, *Baccanella floriformis*, *Duostominidae*, *Endothyridae* etc. and dasycladaceans *Teutloporella nodosa*, *Teutloporella herculea* and other algae. In some places also *Daonella pichleri* and *Posidonia wengensis* have been found (RAMPNOUX, 1974; MARKOVIC & PANTIC, 1975; SUDAR, 1986).

Ladinian Basinal Porphyrite-Chert Formation (Zlatar, Ceotina) graded into basinal Upper Triassic and Liassic sequences, formed by thick and thin bedded limestones and cherty limestones, with intercalations of resedimented calcarenites and microbreccias. Radiolarians, sponge spicules, conodonts and pelagic pelecypods occur in micritic limestones (*Halobia* in Carnian and Norian levels). The resedimented calcarenites and microbreccias contain fossils resedimented from adjacent carbonate platform areas.

In the Mihajlovici zone, however, carbonate platform facies developed during Late Triassic and Liassic; these facies ended with Liassic drowning, when oceanization of internal dinaridic domains started (Jurassic Diabase-Chert Formation, CIRIC, 1954, 1984). The sedimentation ended during the Earliest Cretaceous.

Upper Triassic. In the Carnian shallow water carbonate sequence of Mihajlovici zone the following fossils have been found: *Clypeina besici*, *Poikoporella*, other dasyclads, *Turrispirillina minima*, *Trocholina multispira* and other forams. In the Norian *Megalodon* spp., *Angulodiscus tenuis*, *A. gaschei*, *A. communis* etc. occur and in the Rhaetian: *Triasina hantkeni* and *Involutina turgida*.

Norian-Rhaetian sediments grade upward to Liassic crinoidal-brachiopodal limestones and Upper Liassic Ammonitico Rosso limestones (*Hildoceras bifrons* zone).

4.2.2. North Durmitor-Albanian Alps Carbonate Platform

Durmitor is used by French Authors (BLANCHET, CADET, CHARVET, RAMPNOUX - 1969) to denote a unit with shallow-water sedimentation, contrary to the original use of Durmitor to denote a basin (BESIC, 1948). The compromise name of North Durmitor-Albanian Alps for the carbonate platform, and South Durmitor Basin for the basinal facies domains was suggested by RADOICIC (1982).

Lower Triassic - Mostly reddish to purple micaceous arenites with *Claraia clarai*, *Homomya fassaensis*, *Meandrospira* (Sesian), and thin bedded, impure limestones and marls with *Eumorphotis leptopleura*, *Naticella costata*, *Turbo rectocostatus*, *Tirolites cassianus*, *Meandrospira iulia*, *Ammodiscus incertus* etc. In this area also ooidal limestones occur (Campil).

Middle Triassic - Massive carbonate and brachiopodal limestones with rich foraminiferal-algal association (*Meandrospira dinarica*, *Pilamina densa*, *Macroporella alpina*, *Diploporella hexaster*) passing upward to red nodular encrinuritic limestones (*Paraceratites trinodosus* zone, Han Bulog facies) which grade into the Porphyrite-Chert Formation (according to some authors the Han Bulog itself already carries the first Porphyrite intercalations).

The upper part of Ladinian is characterized by thick (400-500 m) massive, reef type limestones and dolomites in the interior of belt, which at places carries emersion evidences (bauxite). Ladinian reef limestones include prevalingly sponges and algal as well as a rich foraminifer association: *Criptocoelia zitteli*, *C. oenipontana*, *Colospongia catenulata*, *Follicatena cautica*, *Vesicocaulis alpinus*, and also *Teutloporella herculea*, *T. nodosa*, *Bacinella ordinata*, *Baccanella floriformis* and *Duostomina alta*, *Variostoma exilis*, *Endothyranella bicamerata*, *E. pentacamerata* etc. (PANTIC, 1967, 1975).

Upper Triassic - The Upper Triassic is represented by thick foraminiferal-algal and megalodont limestones (cyclothemic arrangement is typical in this part of the Triassic sequence). Laterally also reef like limestones of Dachstein type develop with *Angulodiscus gashei gashei*, *A. tenuis*, *Aulotortus sinuosus*, *Galeanella tollmanni*, *G. panticae*. A very thick Triassic sequence of this unit also outcrops toward SE in Komovi-Visitor area, near the Northern boundary of Albania.

Upper Triassic-Liassic carbonates (Lofer type cyclothem and Dachstein) grade upwards into shallow-water dark-grey and greyish limestone and dolomitized limestone with or without chert, then in the Upper Liassic nodular Ammonitico Rosso or analogous limestones follow.

Dogger-Malm - In continuation with Liassic, different shallow-water Dogger-Lower Malm limestones developed over large platform areas (with Lagenids, *Globuligerina* sp., *Protopeneroplis striata*, *Cladocoropsis mirabilis* and *Kurnubia* etc.). In some areas however, stratigraphic gaps with bauxites (between Upper Liassic Ammonitico Rosso and lagoonal limestones with *Clypeina jurassica*) occur.

Upper Malm-Earliest Neocomian - In the Malm there may be recognized areas with facies of interior platform (characterized by *Clypeina jurassica* and "aberrant Tintinnids") and marginal areas with corals, *Ellipsactinia*, other Hydrozoans, Calcisponges, Nerineids and *Archacosepta basiliensis*, *Pseudocyclammia lituus*, algae and other different algal-like structures.

During the Late Malm and earliest Neocomian, the carbonate platform was gradually reduced and covered by uppermost Malm-Early Neocomian flysch-type deposits with *Calpionella alpina*, *C. elliptica*, *Calpionellopsis oblonga*, *Tintinnopsella longa*, *T. carpathica*.

4.2.3. South Durmitor Basin, (Bosnian zone after the French Authors, Durmitor Flysch after BESIC, 1948).

Triassic and Jurassic sediments do not crop out widely. Triassic succession consists of well known Werfenian facies (more pelitic and more calcarenitic in this upper part), of Middle Triassic Porphyrite-Chert Formation, sometimes with very thick volcanic intercalations, and of Upper Triassic massive limestones interbedded with thick beds of cherty limestones and dolomitized limestones, grading into Lower Liassic.

Jurassic sequences are not sufficiently known and mainly consist of bedded limestones with or without chert, only some tens of metres thick.

Cretaceous - In the uppermost Malm-Lower Neocomian terrigenous (flysch) deposits are common. Upwards, the Cretaceous is represented by very thick resediments that include coarse breccia bodies. They show significant differences in thickness, breccia intercalations, stratigraphic gaps by intense submarine erosion and collapse, and olistoliths (large olistolithes of Porphyrite-Chert Formation have also been observed at Gornja Moraca).

Paleogene - Sediments of this age occur in continuity over the basal Cretaceous succession.

4.2.4. Dinaric Carbonate Platform (High Karst zone, Fig. 8)

This is the largest unit among the Mesozoic domains of Dinarides and its stratigraphy is quite well known. The general evolution of the sequence shows minor facies differentiation during Early Triassic (Fig. 8), and major differentiation during Middle Triassic.

The Upper Triassic is quite homogenous (early stage of carbonate platform development). In the further geologic history important paleogeographic events have to be mentioned at Early Liassic, latest Liassic-Early Dogger, Middle Malm, Middle and Late Cretaceous times (see over).

Lower Triassic (fig. 9, 12) The most diffused sediments are: conglomerates, varicoloured micaceous sandstones, sandy-marly-dolomitic limestones, claystones, marls and ooidal or oncolitic limestones.

In the Sesian, fossils are scarce and represented by *Claraia clarai*, *C. aurita*, *Eumorphotis venetiana*, *Homomia fassaensis* and by foraminifers (*Rectucornuspira kahleri*, *Earlandita tintinniformis* etc.).

In the Southern Montenegro (Crmnica, Brcela) the Sesian ammonites belong to the Asian Province: *Meekoceras vetustus*, *M. gracilitatis*, *Hedenstroemia hyattii* and *Pseudosagoceras multilobatum* (PETKOVIC and MIHAJLOVIC, 1935).

In the Cumpil limestones the fossil content is mostly represented by molluscs (*Naticella costata*, *Myophoria costata*, *M. laevigata*, *Turbo rectocostatus*), foraminifers (*Meandrospira iulia*, *Ammodiscus incertus*, *Glomospira*, *Glomospirella* etc.), and in some places also by ammonites (*Tirolites haueri*, *Dinarites dalmatinus* etc.).

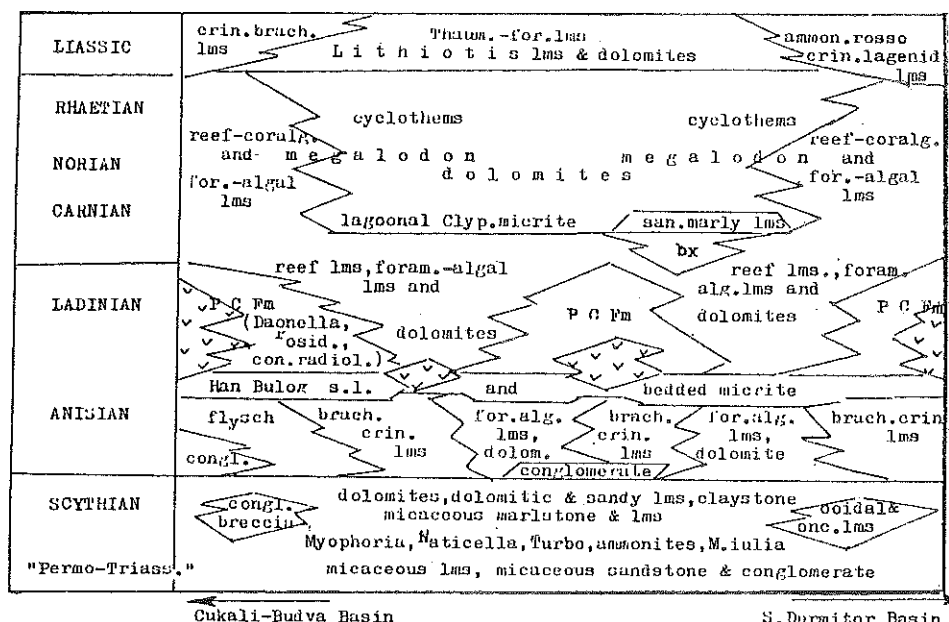


Figure 8. Dinaric Carbonate Platform. Triassic and Liassic stratigraphy. vvv indicate volcanism.

Middle Triassic - Thick terrigenous ("Flysch") sediments (Fig. 9,3) and widespread foraminiferal- algal and crinoidal -brachiopodal limestones characterize the Lower and Middle Anisian in the SW (fig. 9,3a), while during the Upper Anisian Han Bulog facies or their lateral equivalent (nodular or platy limestones) developed (Fig. 9,4). The Ladinian is characterized by the Porphyrite-Chert Formation or laterally by well bedded limestones (Fig. 9,5) to which follow shallow-water limestones and dolomites (Fig. 9,5a). Emersion is at places documented by bauxites. An angular unconformity between Porphyrite-Chert Formation and Carnian limestone is observed in the Rumija region, locally evidenced by slumping.

Upper Triassic - Homogeneous and monotonous carbonate platform facies develop during Late Triassic-lowermost Liassic with cyclothemic dolomites and limestones in the inner zones (Fig. 9,6) and Dachstein - type carbonates with peri-reef facies (Fig. 9, 6a).

A rich fauna, mostly molluscs, is known from Raibl (Carnian) beds (marls, sandy marly limestones, marls and dark-blackish shaly limestones). In the surrounding of Niksic and Trebinje, the following forms have been reported: *Cuspidaria gladius*, *Myophoria kefersteini*, *Avicula kokeni*, *Pachycardia rugosa*, *Trigonodus carniolucus* etc. (PANTIC, 1956, 1957).

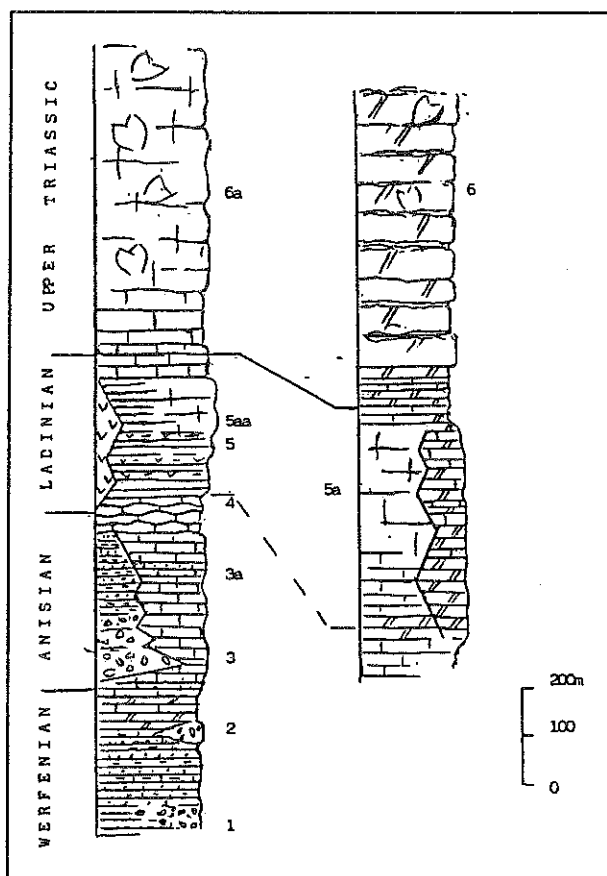


Figure 9. Triassic stratigraphy of the Dinaric Carbonate Platform (D'ARGENIO et al. 1971) in Southern Montenegro (for further explanation see text).

Toward the NW in the Norian-Rhaetian, the Dinaric Carbonate Platform shows some thickness variations in the sequences (probably connected with rejuvenation of the Permian Split-Karlovac transversal line).

Jurassic - In the Jurassic times both marginal and interior carbonate platform facies are known.

Liassic - Over large interior platform areas *Lithiotis*, *Orbitopse*, *praecursor* and *Palaeodasycladus mediterraneus* limestones are diffused. The Liassic platform margins were characterized by coral-spongiomorph-sphinctozoan patch reefs (in which sometimes also lithiotids are observed). Ramps characterized by knoll reefs are associated to deeper crinoidal, brachiopodal and ammonitic limestones; in adjacent basins restricted sedimentation also occurred (Fig. 4).

Dogger - Reduction in size of the platform areas occurred during upper Liassic and early Dogger times, when lagoons were widely overlain by a system of ooidal bars (with some patch reefs) that continued into Malm representing a very productive source of ooidal clastics for both adjacent areas.

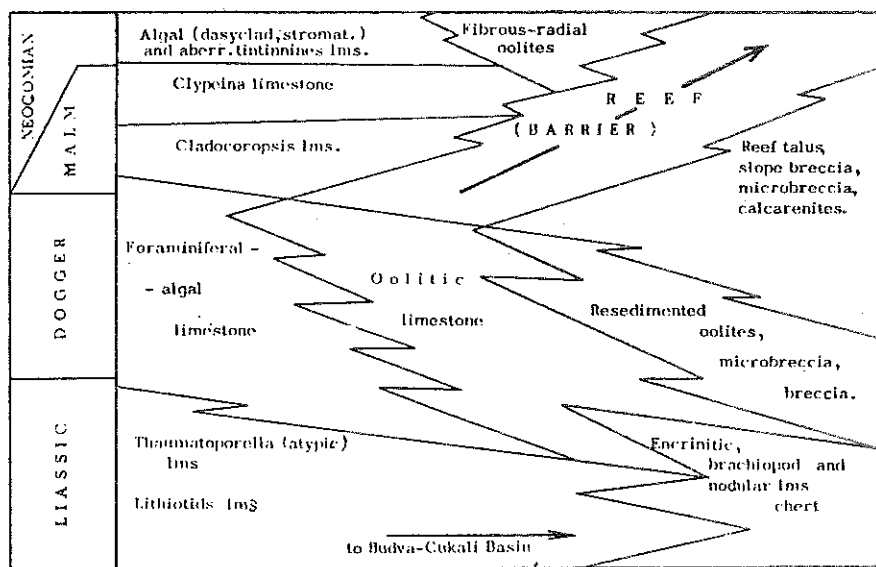


Figure 10. Facies distribution during Jurassic times in the Dinaric Carbonate Platform: the figure shows from left to right, the changes from the Old Montenegro belt toward the Cukali-Budva Basin. Notice the migration of the marginal area with oolitic shoals during the Dogger, and reef development during Malm - Lowermost Neocomian times. Dolomitization is not indicated, but is diffused in the region.

Malm - The beginning of the Malm is significant for the formation of the first large reef complex (coral- hydrozoan- diceratid- nerineid- algal limestones). The migration of the reef complex from the Early Malm into Neocomian suggest a new carbonate platform expansion (Fig.10): it reached a width that was nearly equal to that of the Liassic platform.

During the Malm in the interior platform, developed limestones with *Cladocoropsis mirabilis*, *Paraurgonina caelinensis*, *Kurnubia jurassica*, *Clypeina jurassica* and fibrous-radial ooids. Brackish-water episodes developed also in the Malmian platform interior sequences (Upper Kimmeridgian). Significant stratigraphic gap between Liassic and Upper Malm, and shorter one in the Malm, affected part of the platform interior (often evidenced by bauxite), (Figs. 5, 11).

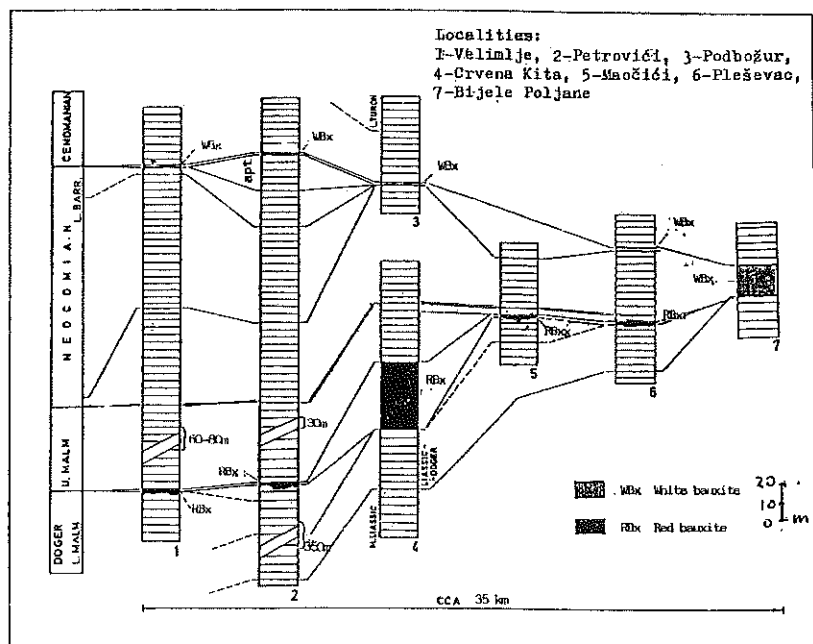


Figure 11. Jurassic and Lower Cretaceous stratigraphic gaps and bauxite horizons in Old and West Montenegro (from RADOICIC and VUJISIC, 1970, simplified).

Cretaceous - In the Cretaceous shallow water sequences of platform interior are very diffused.

Neocomian. Stromatolitic, mainly dolomitized limestones with gastropods, ostracods, *Cyanophyta* and dasyclads, are diffused.

Barremian to Albian. Different algal, algal-foraminiferal and foraminiferal limestones and limestones with small nerineids, dasyclads and foraminifera (*Actinoporella podolica*, *Salpingoporella muhelbergi*, *S. melitae*, *S. cemi*, *S. dinarica*, *S. turgida* and *Orbitolinidae*, *Debarina*, *Sabaudia*, etc.) occur.

In a limited area of Montenegro and in some areas of Hercegovina (Fig.5) *Charophyta* limestones and marly limestones may also be found in some Lower Cretaceous levels.

Cenomanian. Limestones with *Chondrodonta* and other *Ostreidae* (fig. 12) and limestones with *Ovalveolina*, *Sellialveolina*, *Praealveolina*, *Cisalveolina fraasi*, *Cuneolina*, *Soritidae* and the algae *Heteroporella lepina* and *Neomeris*, hydrozoans, bryozoans etc.

Turonian and Senonian. Limestones and dolomitized limestones with *Aeolisaccus kotori*, rare foraminifers, discorbids, ostracods and rudists.

Middle-Upper Senonian. Limestones with *Accordiella conica*, *Dicyclina schlumbergeri*, *Scandonea*, *Moncharmontia*, and rudists;

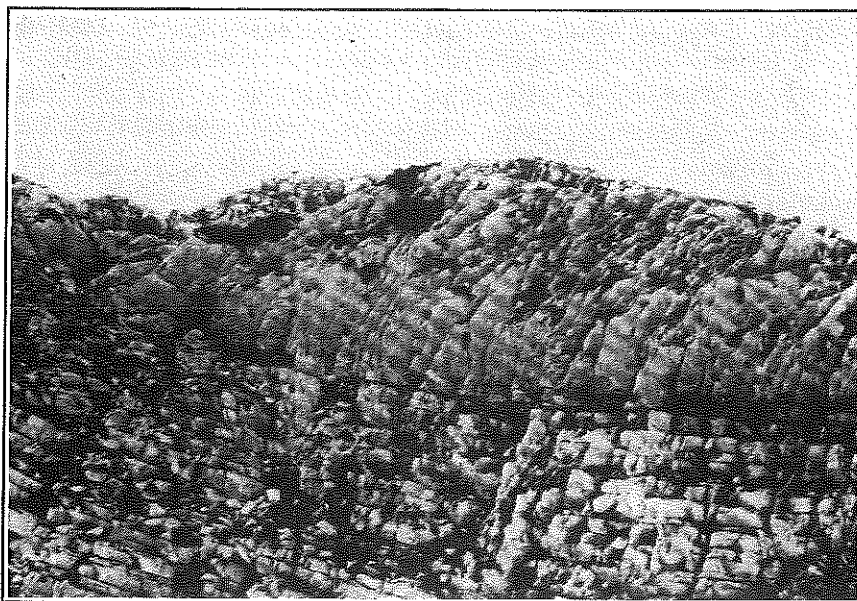


Figure 12. Cenomanian thin bedded *Chondrodonta* limestones covered by *Caprinid* massive limestones in Vukotina Golija Mt.- area (from VUJISIC, 1974).

Maastrichtian. Limestones and dolomitized limestones with *Rhytidionina liburnica* and *Laffiteina mengaudi*.

Platform margin facies. Like in Italy, different types of margins are formed by sediments deposited in a number of subenvironments alternating each other along a belt of high turbulence (reef, reef breccias and calcarenitic shoals). The period of highest reef growth was from Early Malm to Lowermost Cretaceous. While Malmian reefs are generally well preserved, Cretaceous reefs may be found only as blocks in more recent Paleogene and Miocene sediments.

Neocomian reefs were formed by hydrozoan, calcisponges and algae (*Tubiphytes*, *Lithocodium*, *Bacinella*, *Koskinobullina*, *Mercierella*, different algal and algal-like structures, *Pseudocymopolia jurassica*, *Macroporella praturtoni*) and by foraminifers (very common are *Protopeneloplis trochangulata*, *Trocholina*, *Pseudocyclamina*).

Barremian and Aptian reefs were built by colonial corals, calcisponge, molluscs (caprinids, requienids, nerineids) with a noticeable contribution of dasyclads and other algae.

During the Upper Cretaceous different rudistid prevailing patch reefs were distributed (caprinid reefs in Cenomanian, Fig. 12, *Hippuritidae* *Radiolitidae* in Upper Senonian).

Late Cretaceous-Early Tertiary emersions and drowning. In part of the Dinaric carbonate platform drowning related pelagites were deposited during

Turonian or Turonian-mid Senonian times; while in the area of Zeta furrow pelagic sediments were deposited from Upper Senonian through Paleogene. However over large sectors of the Dinaric Platform at the end of Cretaceous, after widespread stratigraphic gaps, shallow-water Paleocene to Middle Eocene limestones follow.

AGE	PALAEOTECTONIC STAGE	STRATIGRAPHIC UNIT	LITHOFACIES	SEDIMENTARY ENVIRONMENTS
MIocene	DROWNING	TERRIGENOUS	MARLSTONES, SANDSTONES, SHALES, FORAMINIFERAL LIMESTONES AND STROMATOLITES	SCARP AND BASIN
LOWER TERTIARY			GAP BAUXITES	OPEN LAGOON AND TIDAL FLAT
UPPER CRETACEOUS	EPIOCEANIC (MARGINAL) CARBONATE PLATFORM	CARBONATES	RUDIST LIMESTONES, OSTREID & FORAMINIFERAL LIMESTONES	INNER CARBONATE PLATFORM AREAS
LOWER CRETACEOUS			ALGAL LIMESTONES, FORAMINIFERAL LIMESTONES, DOLOMITES	
MALM			ALGAL, FORAMINIFERAL, FIBROUS OGDAL LIMESTONES AND DOLOMITES	
DOGGER	INTRACONTINENTAL EVAPORITIC CARBONATE PLATFORM	NOT	CROPPING OUT	SABCHA, TIDAL FLAT, OPEN LAGOON
LIAS				
UPPER TRIAS				
MIDDLE TRIAS	DEFORMATION ?	PORPHYRITE		
LOWER TRIAS				

Figure 13. Adriatic Carbonate Platform, southern sector: stratigraphic scheme from Triassic to Miocene time.

4.2.5. Adriatic Carbonate Platform (Fig. 13)

The carbonate deposits belonging to the Adriatic Platform Domain outcrop along the Adriatic coast of Montenegro and Croatia. A large part of the southern platform segment is submerged in the present day Adriatic Sea. Such a situation makes it difficult to identify its southwestern rim (which is recognized in Susac Island for the Malm-Neocomian interval).

The contact with tectonic imbricates of more internal origin also hampers to recognize the northern margin. The only outcrops attributable to the northern rim of the Adriatic Carbonate Platform Domain are the reef-like limestones of Upper Senonian to the north of Peljesac peninsula.

The sediments representing the central part of this domain outcrop widely along the Adriatic coast of Montenegro and southern Dalmatia, including part

of the Dalmatian Archipelago. In general the facies pattern is similar to that of the Dinaric domain.

4.3 Selected sequences of the Southern Dinaric Carbonate Platform.

4.3.1 North-Eastern Margin.

The following sequences are typical of the North-Eastern margin of the Dinaric Carbonate Platform domains.

Gacko sequence (Fig. 14 A)

- Common rocks are fine grained greysh dolomites with intercalation or rare nodules of chert, micritic limestones with very rare foraminifers (*Ophtalmidium*, *Glomospira*, lagenids), sponge spicules and some radiolarians; fine grained greysh dolomite with cherty intercalation (Lower Liassic);

- Greysh to dark grey limestones, silicified and dolomitic limestones with very rare fossils: foraminifers (*Ophtalmidium*, *Glomospira*, *Involutina*, lagenids), sponge spicules, *Stomiosphaera* and radiolarians (Lower-Middle Liassic);

- Grey to blackish platy to thicker bedded limestones, at places shaly and silicified, somewhere with Domerian ammonites (GAKOVIC, 1986);

- Bedded and thick bedded limestones and dolomitized limestones with *Ophtalmidium*, *Glomospira*, *Involutina*, lagenids, *Aeolisaccus*, sponge spicules and radiolarians (Upper Liassic);

- Bedded black marly limestone with ammonites (Lower Dogger GAKOVIC, 1986);

- Breccia with elements of Liassic and lowermost Dogger age, ooidal and bioclastic calcarenites and microbreccias (*Protopeneroplis striata*) and calcilutites with chert intercalations and chert nodules (Dogger-lowermost Malm);

- Reef complex limestones including scarp breccias (Malm- lowermost Neocomian).

- Coarse scarp breccias with Lower and Middle Cretaceous shallow-water and pelagic elements, passing upward into resedimented carbonates (breccias, microbreccias, calcarenites), calcilutites, thin bedded limestones and marly limestones with chert intercalations and conglomerate (Upper Cretaceous to Middle Eocene).

This part of the platform marginal area had a complex history during the Jurassic and Cretaceous: due to platform progradation a reef complex developed over part of Zalomka-Gacko adjacent basin, during the Latest

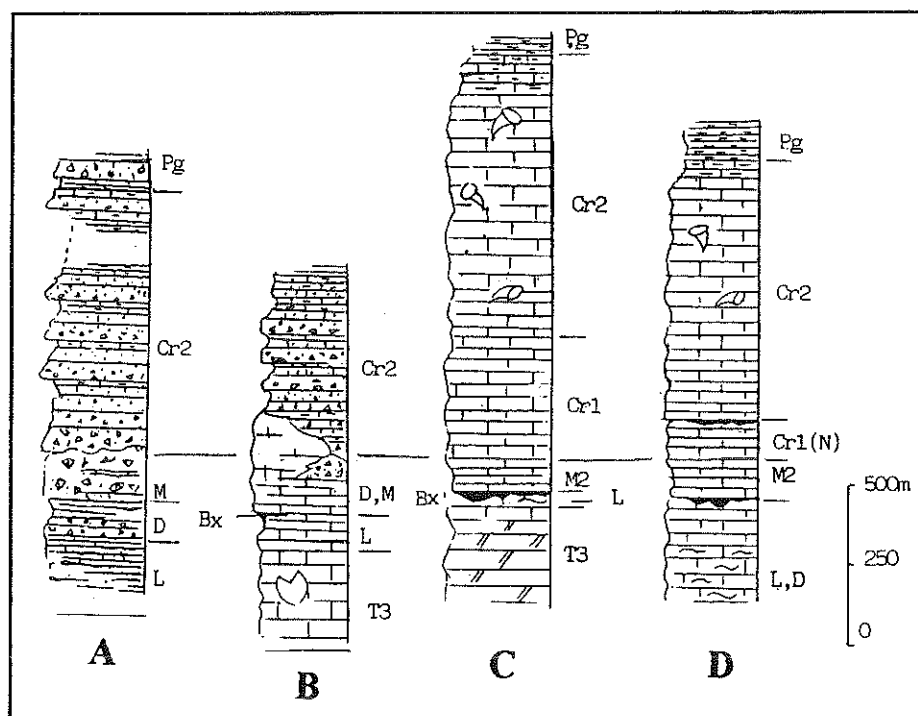


Figure 14. Stratigraphic columns of the Dinaric Carbonate Platform (from D'ARGENIO, RADOICIC & SGROSSO, 1971).

A: Gacko; B: Vojnik; C: NE Niksic; D: SW Niksic

T3 – Upper Triassic dolomites and limestones; L – Liassic, a: euxinic with ammonites, b: Ammonitico Rosso, c: with Lithiotis; LD – Liassic (with Lithiotis) and Dogger carbonates; D – Gacko: limestones with Early Dogger ammonites, breccia, ooidal and bioclastic calcarenites and calcilutites with chert; DM – Vojnik: ooidal calcarenites, *Clypeina jurassica* beds and reef complex limestones; M – Gacko: forereef breccia; M2 – *Clypeina jurassica* beds; Cr1 – shallow water platform interior carbonates, Cr1(N) – Neocomian beds; Cr2 – breccia and carbonatic turbidites (Gacko and Vojnik), rudistid limestones (Niksic); PG – carbonate clastics, marls and silty limestones; Bx – bauxites.

Dogger-Malm, while at the beginning of Late Cretaceous the same area passed into basinal domain (basin widening process).

Mountain Vojnik (Fig. 14B)

- Back reef dolomites and *Megalodon* limestone, limestone with *Triasina hantkeni*; short stratigraphic gap, followed by limestone with very rare *Lithiotis* (Upper Triassic-Liassic).

- Ammonitico Rosso: reddish marly nodular limestones and red limestones with ammonites, (*Hildoceras bifrons*), Liassic;

- Microbreccias and limestones with *Globuligerina* and pelagic pelecypods (Dogger)
- Ooidal calcarenites, calcarenites with *Protopeneroplis striata*, *Archaeosepta basiliensis*; reefs with *Ellipsactinia*, nerineids, corals etc. (Dogger – Lower Malm).
- Platform limestone with *Charophyta* and *Salpingoporella grudii*, limestones and dolomites with *Clupeina jurassica* and fibrous-radial ooids. (Upper Malm-lowermost Neocomian).

4.3.2. Central Domains of the Platform

The central part of the Dinaric Carbonate Platform generally has characters of inner lagoon. Some variations may be nevertheless observed in the following three sequences:

North-East Niksic (Fig. 14C)

This area belonged, from the end of the Cretaceous, to the intraplatform Zeta basin.

A large stratigraphic gap characterizes this sequence in which Upper Malm to Upper Cretaceous lagoonal limestones and dolomites overlap Triassic to Lower Liassic limestones and dolomites (with or without bauxites over the latter rocks). In the NE Niksic sequence, calcilutites (Upper Senonian) and marls (Paleogene) with pelagic foraminifers and nannofossils developed.

South-West Niksic sequence (Fig. 14D)

Niksic south-western sequence belongs also to the Zeta furrow, it differs from the previous one because two stratigraphic gaps evidenced by bauxites may be recognized: between Lower Dogger and Upper Malm, with red bauxite, and between Neocomian and Cenomanian (or Albian-Cenomanian) with white bauxite.

It must be mentioned that in two localities south of Niksic an interesting Liassic and Neocomian fossil flora has been found:

(a) in the north slope of Mount Budos the Liassic flora is composed of *Equisetites* cf. *columnaris*, *Philophylum pecten*, *Widringtonites liasinus*, *Podozamites lanceolatus*, *Lomatopteris schimpheri*, *Otozomites* sp. and *Chladophlebis denticulata* (PANTIC, 1952);

(b) in Carev Most (E of Budos) the Neocomian flora includes *Elatides* sp., *Sphenolepidium sternbergianum*, *Podocarpus mucronata* and *Pinus* sp. (PANTIC, 1958).

Old Montenegro (Fig. 16 A)

The Old Montenegro zone has typical interior platform facies which extend from the Zeta plain toward South Hercegovina and Biokovo Mountain.

The differences from the previously described sequences are mostly in the higher thickness of the sediments and in the apparent lack of Jurassic stratigraphic gaps, although some bauxites evidence at places an emersion during part of the Kimmeridgian.

The sequence is represented by:

- Upper Triassic-Lower Liassic stromatolitic dolomites and limestones (cyclothems) which in the Carnian-Norian levels have, at places, a thin intercalation of coal (Prelaka, Nudo);
- Jurassic and Cretaceous facies of interior-lagoonal platform;
- Upper Cretaceous limestones and rudistid limestones;
- Short gap in the Upper Maastrichtian-Lower Paleocene;
- On the Cretaceous sediments Paleocene- Eocene limestones finally follow.

Dinaric Intraplatform furrows.

Several bauxite outcrops are known in the Old and Western Montenegro (Fig. 11). This is specific area of white bauxite occurrence in which some large stratigraphic gaps (e.g. Liassic-Cenomanian, lowermost Neocomian-Cenomanian) appear as being due to repeated karstic events (RADOICIC, 1982).

As already mentioned, the sequences, in the NE and NW Niksic area belong to the Zeta intraplatform furrow. One of the characteristics of Dinaric Carbonate Platform is the development of intraplatform hemipelagic facies during Latest Dogger-Middle Malm (Lemes furrow, Fig. 15) and Upper Cretaceous up to Miocene (Zeta furrow).

The Zeta furrow is directly connected with the adjacent Zalomka-Gacko basin (Liassic-Lower Malm, at places up to Aptian-Albian), which deeply penetrates the platform, (RADOICIC, 1987; Fig. 4). The origin of these basins may be related to the synsedimentary tectonic activity during the Mesozoic.

4.3.3. South-West margin of the Platform .

The southwestern rim of the Dinaric Platform outcrops from the Rumija Mountain to the Lovcen and North Boka areas. Two sections (western Rumija and North Boka) well represent the stratigraphic characters of this margin. Besides North Boka, here it is worth mentioning another section with megabreccia from Neretva region (Vukov Klanac), because in this succession megabreccias appears in more internal position.

Western Rumija (Fig. 16B)

The Western Rumija sequence is formed by:

- Thick back-reef dolomites (Late Triassic-lowermost Liassic);
- Open platform to patch reef facies (Liassic); and crinoid -lagenid limestones.
- Ooidal bioclastic limestones (Dogger); reef and perireef limestones (Lower Malm);
- Lagoonal limestones and dolomitized limestones with *Charophyta*, *Paraurgonina* and *Clypeina jurassica* (Upper Malm) grading into lagoonal sediments (Neocomian).

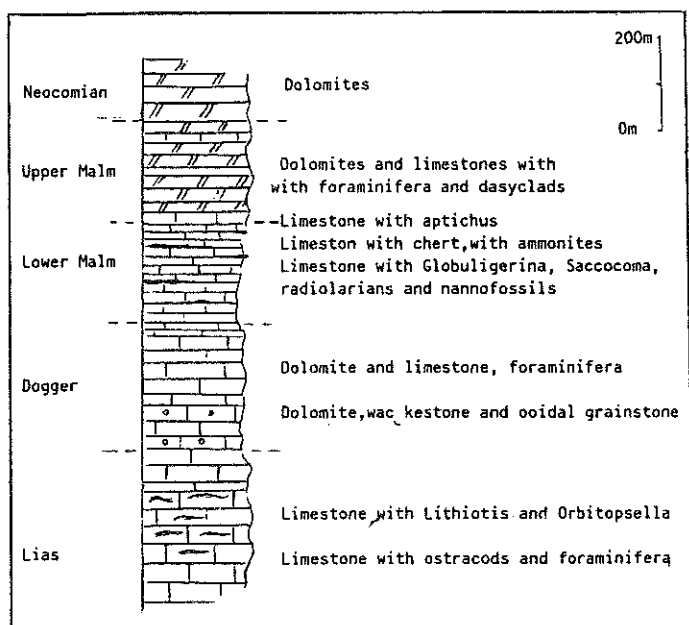


Figure 15. Lemes stratigraphic column (Lower Malm intraplatform furrow).

North Boka (Fig. 16C)

North Boka sequence consists of:

- Reef complex limestones (Upper Triassic-lowermost Liassic);
- White and reddish calcilutites and calcarenites with ammonite embryos and lagenids (Liassic);
- Calcarenites, reef and reef breccias with rare interfingering pelagic calcilutites with Tintinnids (Malm to lowermost Cretaceous)
- Stratigraphic gap (due to submarine erosion?)
- Megabreccia with intercalations of fine grained bioclastic or foraminiferal limestones in the lower part, and then micritic limestone and greenish marly limestones with pelagics (Paleocene to Middle Eocene)
- Upper Eocene to upper Miocene terrigenous deposits.

Vukov Klanac, Neretva River (Fig. 16D).

In the surroundings of Vukov Klanac the sequence is formed by:

- Dolomites and limestones (at places silicified), in banks and thick beds with very rare *Lituosepta*, *Palaeodasycladus*, *Pachymegalodus* (Uppermost Triassic-Lowermost Liassic and Lower-Middle Liassic). In the Liassic sequence breccias (due to Liassic tectonics?) are present;
- Megabreccia, with Upper Cretaceous clasts occur above the Jurassic and then at the topmost of the sequence there are reddish hardgrounds with *Globotruncana* (uppermost Cretaceous);

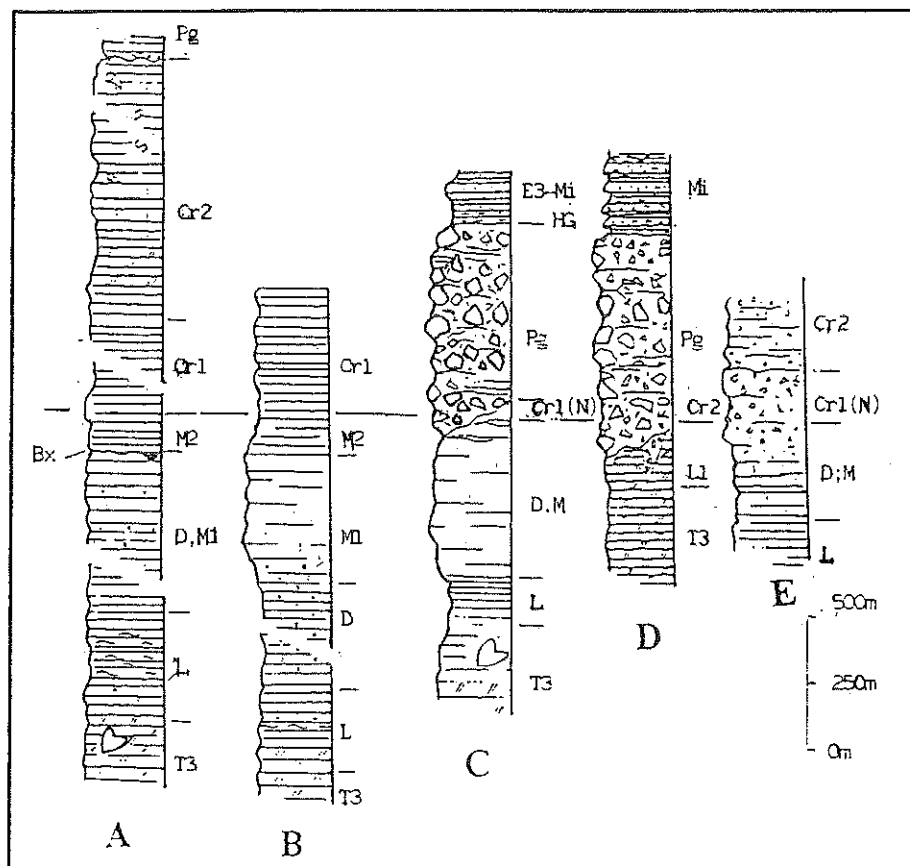


Figure 16. Stratigraphic columns of the Dinaric Carbonate Platform and transition to the Cukali-Budva Basin (from D'ARGENIO, RADOICIC & SGROSSO, 1971). **A**, Old Montenegro and **B**, Central Rumija: T3-Upper Triassic dolomites and limestones; L-Liassic dolomites and limestones with Lithiotis; D M1-Dogger and Lower Malm platform interior facies; D-Dogger ooidal limestones; M1-Lower Malm reef facies; M2-Upper Malm lagoonal beds with *Clypeina jurassica*; Cr1-Lower Cretaceous platform interior facies; Cr2-Upper Cretaceous dolomites and rudistid limestones; Pg-Paleogene breccias and carbonates. **C**, North Boka (slope to basin facies): T3-Upper Triassic dolomites and limestones; L-Liassic dolomites and limestones with Lithiotis; D,M-Dogger-Malm perireef facies; Cr1 (N)-Neocomian reef-perireef carbonates; Pg-Paleogene breccias and carbonates; HG-hardground; E3-Mi-Eocene-Upper Miocene terrigenous. **D**, Vukov Klanac: T3-Upper Triassic dolomites with cherts in the topmost part; L1-Lower Liassic dolomites and limestones and breccias; Cr2, Pg, Mi3-Uppermost Senonian-Paleogene-Upper Miocene megabreccias, carbonate clastics and terrigenous. **E**, Djienas: L-hemipelagic partly dolomitized liassic limestones; D;M-Dogger-Malm silicified limestones, ooidal calcarenites, microbreccias, silicified marls and organogenous breccias; Cr1(N)-Neocomian massive breccias; Cr2-Upper Cretaceous massive breccias with rudist fragments and planktonics in micritic matrix..

- Megabreccia, breccia with greenish sandy marly matrix mixed with terrigenous sediments with pelagic foraminifers and nannofossils (Paleogene-Miocene), then follow.

4.4. Selected sequences of the Cukali-Budva Basin (Fig. 17).

The sediments of this basinal domain range in age from Lower Triassic to Upper Miocene (in the Boka Kotorska region from the Ladinian). Typical sections outcrop in the Budva and Boka Kotorska region.

At present the basinal Cukali-Budva sequences are interposed between the Dinaric and the Adriatic platform domains only from Skadar to the West Boka Kotorska areas. To the Northwest, the Dinaric Platform rocks are thrust directly over the Adriatic Carbonate Platform sequences.

4.4.1. Boka Kotorska Region (composite sequence).

A typical sequence is given by:

- Turbiditic sandstones, arenaceous pelites, marls and marly limestones (Anisian);

- Porphyrite-Chert Formation composed by porphyrites, tuffs, bedded cherts, more or less silicified argillites, marls and limestones (Ladinian) passing upward to

- Bedded cherty limestones with pelagic pelecypods (*Halobia*) prevailing in the central part of the belt; calcilutites, resedimented carbonates and intraformational breccias in the basin marginal sectors (Ladinian-Liassic).

- Calcilutites with chert nodules and lens intercalations, siliceous limestones, argillites, radiolarites and jaspers prevailing in the central part of the belt; calcilutites with chert intercalations and resedimented calcarenites and carbonatic breccias of variable grain size, turbiditic carbonates along the marginal parts of the belt (Liassic-Upper Cretaceous);

- Pelagic, scaglia-type, limestones in the central part of the belt, then resedimented bioclastic turbiditic carbonates with chert intercalations and more or less silicified limestones; along the sides of this basinal belt coarse to very coarse breccias increase dramatically;

- Red marls, marly limestones, sandy marls, siltstones, calcarenites and terrigenous resediments in the Paleogene and Miocene.

From the platform margin facies toward the basinal facies, the scarp sediments are represented by different rocks whose lithology is related to the type of facing platform: as this margin may change with time, the basin deposits show a vertical variation and repeatedly intercalate in the platform scarp deposits. The most common deposits are resedimented calcarenites and calcirudites, often well graded, with a conspicuous amount of bioclastic resediments and bedded chert development. A lateral (partly more distal) equivalent is represented by cherty "calcilutites" (often made of biogenic components which particularly in the Upper Jurassic and Cretaceous are represented by nannofossils). Moreover, the sequences of Cukali-Budva

originally located near the platform margins show very different stratigraphic organization, even at very short distance from each other.

AGE	PALEOTECTONIC STAGE	STRATIGRAPHIC UNIT	LITHOFACIES	SEDIMENTARY ENVIRONMENTS
MIOCENE		FLYSCH	SANDSTONES, SILTSTONES, SHALES	TERRIGENOUS BASIN
LOWER TERTIARY			RED SILTSTONES & MARLSTONES	
UPPER CRETACEOUS	EPIOCEANIC (MARGINAL) BASIN	SCAGLIA	PLANKTONIC FORAMINIFERAL & CALCILUTITES	
LOWER CRETACEOUS		CALCAREOUS-SILICEOUS	CHERT Limestones	
MALM		& CLASTICS	CALPIONELLID & CHERT Limestones	
DOGGER		SEDIMENTS	CLAYSTONES, CHERT Limestones & SILICIFIED CLAYSTONES & CLASTICS	
LIAS	INITIAL CONTINENTAL BASIN		CHERT Limestones	
UPPER TRIAS			REDEPOSITED SLAP BRECCIA	
MIDDLE TRIAS		POPHYRYT CHERT FM.	ANDESTITIC POPHYRY, PILLOW LAVAS, GREEN TUFFITES, CHERT Limestones WITH PLANKTONIC MOLLUSCS, REEF Limestones (LADINITIAN)	
L. TRIAS		TRIASSIC FLYSCH	SANDSTONES, SHALES, CONGLOMERATES	TERRIGENOUS BASIN

Figure 17 - Cukali-Budva Basin. Stratigraphic scheme from Triassic to Miocene times.

4.4.2. Djeanas sequence (platform-basin transition, Fig. 16E)

Closely associated with the reefs are more or less coarse reef breccias, passing laterally to microbreccias and calcarenites; at times they are the only evidence of a reef development. These breccias grade to faults scarp breccia produced by active faults along the platform rims. One of this type area is Djeanas, whose section consists of:

- Stratified hemipelagic limestones (partly dolomitized) with very rare nodosarids and some radiolarians (Liassic);
- Silicified limestones, ooidal calcarenites, microbreccias and calcilutites with chert intercalations, in place silicified (upper part), and silicified marls. Calcilutites with radiolarians and sponge spicules; bioclastic calcarenites and microbreccias with *Protopeneroplis striata*, other foraminifers and different metazoan debris (Dogger-Lower Malm);
- Bioclastic strata made of calcarenites, microbreccias and organogenic breccias; upwards passing to a thick, massive interval made of scarp fore-reef breccias with micritic and somewhat recrystallized matrix. Among the bioclasts are: *Ellipsactinia*, and other hydrozoans, calcisponge, corals, *Tubiphytes*, *Koskinobullina*, *Mercierella* fragments, *Clypeina jurassica*, codiacean and

other algae structures, then other foraminifera, among which *Nautiloculina circularis*, *Archacosepta basiliensis* and *Coscinophragma* (Upper Malm);

- Massive breccia with similar contents plus *Dictyoconus* gr. *walnutensis-arabicus*, *Pseudocymopolia jurassica* and rare tintinnids in a micritic matrix (Neocomian);

- Massive breccias with rudistid fragment and *Globotruncana* in the micritic matrix (Turonian ? - Senonian).

4.4.3. Vrmac sequence (Fig. 18A)

In the Vrmac-Verige belt one of the most typical proximal sequences of the Cukali-Budva Basin is outcropping; it ranges from Middle Triassic to Paleogene and is an expression of the northern slope of the basin.

- Reef limestones (sponges, corals, dasyclads, other algae, brachiopods) and breccia (Ladinian) (Fig. 18A-1);

- Calcilutites with chert intercalations, calcarenites and intraformational breccias. Calcilutites with radiolarians, sponge spicules, *Halobia* spp., in the calcarenites *Galeanella tollmanni* and other foraminifers (Upper Triassic, Fig. 18A-2);

- Breccias with lithoclasts of late Triassic age, calcilutites, calcarenites, microbreccias, nodules and chert intercalations - *Involutina farinacciae*, *Involutina liassica* (Liassic, Fig. 18A-3);

- Very coarse breccia with lithoclasts of different Triassic and Liassic age (the latter with *Involutina liassica*), completely silicified calcilutites and ooidal calcarenites with argillitic and cherty levels at the base and graded breccia at the top (Dogger, Fig. 18A-4);

- Alternation of breccia, microbreccia, calcarenite, ooidal calcarenite and calcilutite, argillite at the base, silicified levels, nodules and chert intercalations, in the upper part dolomitized breccia, breccia with white micritic clasts, breccia with thin micritic levels with *Calpionella alpina*. Throughout the interval: *Ellipsactinia* and other hydrozoans, calcisponges, *Tubiphytes*, *Mercierella*, *Koskinobullina*, different algal and algal-like structures occur (Malm – lowermost Neocomian, Fig. 18A-5).

- Breccia, microbreccia, calcarenite the great part of which is totally silicified; the micritic matrix of the lower breccia levels contains *Tintinnopsella carpathica*, *Calpionella elliptica*, *Calpionellopsis oblonga*. Bioclasts are the same as in the Neocomian breccia of Djenas sequence. Upwards follows an alternation of silicified calcarenites, calcilutites, argillites, very thin argillitic levels and chert, marly limestones and calcarenites. In the latter *Orbitolinidae*, *Hedbergella* and *Stomiosphaera* are found (Lower Cretaceous, Fig. 18A-6); It may be noted that Malm-Lower Cretaceous part of Vrmac succession shows a composition close to that of "Fanusi" Dolomite and with the calcareous member of the Crisanti Formation in Sicily (SCANDONE et al., 1972);

- Thick bank of very coarse and graded polygenic breccias (Fig. 18A-7); which upwards may reach the uppermost Senonian.

An abrupt lithological change marks the Cretaceous/Tertiary boundary; marly and sandy, silty marly sediments with *Globigerina eugubina* (Paleocene) follow.

4.4.4. Lastva (Fig. 18B)

The Lastva sequence belongs to the axial part of the Cukali-Budva Basin. The oldest outcropping sediments are reef limestones and reef breccias (Fig. 18B-1) of Ladinian age followed by:

- Limestones, cherts and thin beds made of marls, argillites and tuffites (uppermost Middle Triassic- lowermost Upper Triassic, Fig. 18B-2);

- Chert, cherty and marly limestones with *Halobia superba* and tuffitic levels (Upper Triassic, Fig. 18B-3);

- Limestones, cherts, tuffitic levels, cherty limestones, levels of breccias with tuffitic clasts, cherty marls and argillites with *Halobia beyrichi*, *H. styriaca* (Fig. 18B-4);

- Cherts (this interval is badly exposed due to tectonic deformation) (Fig. 18B-5);

- Limestones with cherts and cherty limestones, silicified marls and argillites; in the upper part are present: *Halobia* sp., and the conodonts *Epigondolella abneptis*, *Epigondolella bidentata*, (Upper Triassic, Fig. 18B-6);

- Chert (not well exposed, Fig. 18B-7), (Upper Triassic);

- Limestones with chert, (at places affected by slumping), thin marly levels, silicified marls and argillites, limestones, marly levels, radiolarites, argillite and cherty argillites, marly limestones (Upper Triassic, Fig. 18B-8);

- Bank of graded microbreccias, calcarenites, calcilutites, argillites, ooidal microbreccia and calcarenite, (Liassic, Fig. 18B-9). In the lowermost part there are bioclasts with: *Thaumatoporella*, *Codiaceae*, crinoid fragments, *Involutina turgida*; in other beds, fragments of molluscs, crinoids, codiacean and dasycladacean algae, *Nodosaridae*, *Textularidae* and other foraminifers occur;

- Alternation of limestones and chert (graded breccias, microbreccias, calcarenites with ooids and calcilutite); in the middle and upper part: *Involutina farinacciae*, *Glomospira*, *Ophtalmidium*, *Lagenidae*, *Mesoendothyra* (Liassic, Fig. 18B.10);

- Alternation of limestones, marls, microbreccias and chert (Liassic or-? lowermost Dogger, Fig. 18B-11);

- About 40 m not exposed: (fault ?) (Fig. 18B-12);

- Radiolarite, few argillites, mainly silicified microbreccia with *Gutnicella cayeuxi* (Dogger, Fig. 18B-13);

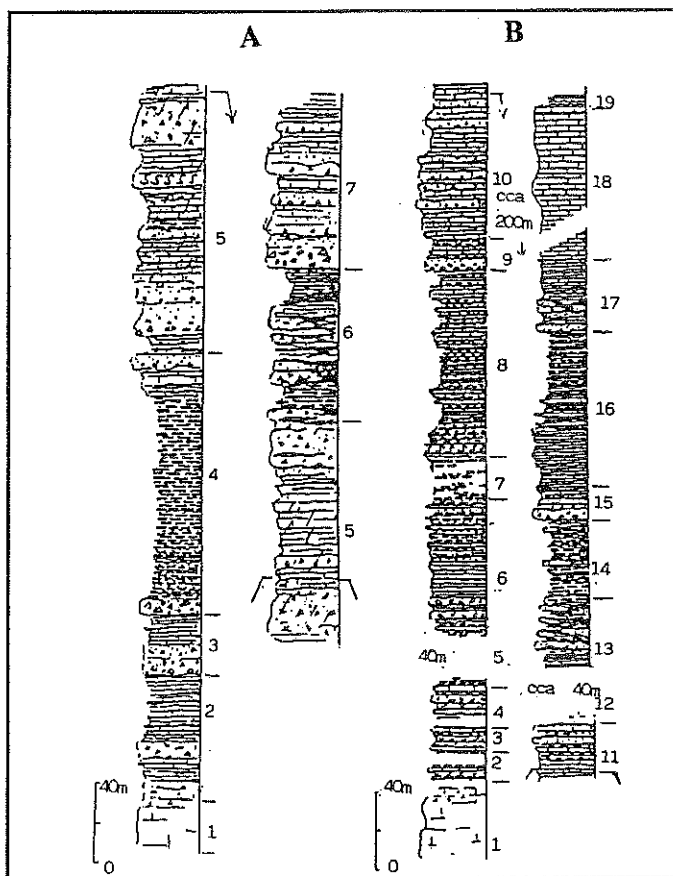


Figure 18. Stratigraphic columns of Vrmac (A) and Lastva (B). For further explanations see text.

Radiolarites, argillites and prevailing silicified breccias, microbreccias and calcarenites with *Protopeneroplis striata* (Dogger-Lower Malm, Fig. 18B-14);

- Breccias, silicified calcilutite, microbreccia and breccia, rare marly and somewhat dolomitized levels. In the clasts: *Tubiphytes morronensis*, *Mercierella dacica*, *Koskinobullina socialis*, *Bacinella irregularis*, in the uppermost part *Calpionella alpina* and nannofossils (Upper Malm – lowermost Neocomin, Fig. 18B-15);

- Thin bedded reddish limestones with cherty lens and bedded cherts, radiolarites with rare limestone levels, and with silicified microbreccias (Lower Cretaceous, Fig. 18B-16);

- Partly silicified breccias, calcarenites and calcilutites, thin red marly levels, with a thin pyroclastic level, bedded cherts. In the breccias and

calcarenites the bioclasts are prevailingly represented by rudistids and orbitolinids. Pelagic components in the lowermost part: *Rotalipora cushmani* and *Praeglobotruncana stephani*, *P. delrioensis*; in the last few metres, *Marginotruncana* (Cenomanian to Turonian or lowermost Senonian, Fig. 18B-17);

- Upwards, a thick sequence of light grey and reddish *Globotruncana* limestones (scaglia) follows. In the topmost Senonian part, thin levels of calcarenites with *Omphalocyclus* and orbitoid fragments also occur. Finally, with an abrupt change, Paleogene calcareous deposits follow (*eugubina* zone).

APPENDIX

A four-day field trip across the Southern Dinarides

First Day

Theme: Mesozoic-Early Tertiary facies domains and tectonic features of the Dinaric coastal belt, from Dubrovnik (Croatia) to Montenegro. Adriatic Carbonate Platform, Cukali-Budva Basin and Dinaric Carbonate Platform.

Route: Dubrovnik to Sutomore (km 220)

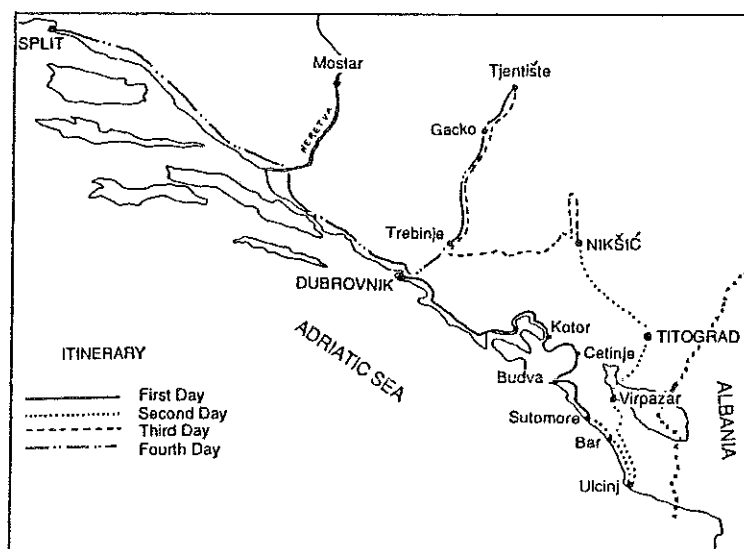


Figure 19. Index map of the field trip

Dubrovnik, among the most beautiful Mediterranean towns, along the Adriatic Coast, is built on Cretaceous limestones pertaining to the Adriatic Carbonate Platform.

The excursion route follows the coast toward the South. After few km we enter in the Zupa Dubrovacka, where it is possible to observe the Dinaric Carbonate Platform strata (represented by Upper Triassic dolomites passing upwards into Jurassic and Cretaceous limestones) thrust over the Adriatic Platform (here formed by Cretaceous to Eocene carbonates and by Paleogene-Miocene terrigenous deposits, Figs. 20, 21).

This thrust line may be followed on the left of the road through the Konavli Polje (where Paleogene-Upper Miocene terrigenous sediments outcrop de CAPOA et al., 1991,1995) down to Boka Kotorska area, with some interruptions due to the transversal Plat-Slivnica line.

On the right there are large outcrops of the Adriatic Carbonate Platform sequences formed by Upper Cretaceous carbonates, up to the Maastrichtian with *Rhypidionina liburnica* and *Laffitteina mengaudi*. In the Boka area the

Cretaceous is topped by sparse bauxite lenses over which thin Paleogene carbonates follow.

Moving from Konavli toward Hercegnovi and at the boundary with Crna Gora (Montenegro) we approach Boka Kotorska bay, the submerged valley of the Boka River (CVIJIC, 1924). Here we may see the first Cukali-Budva sequences.

From Bijela and Sveta Nedjelja spur the road faces the sea shore and along the road pelagic cherty limestones with *Halobia* (upper Triassic) outcrop (Fig. 22).

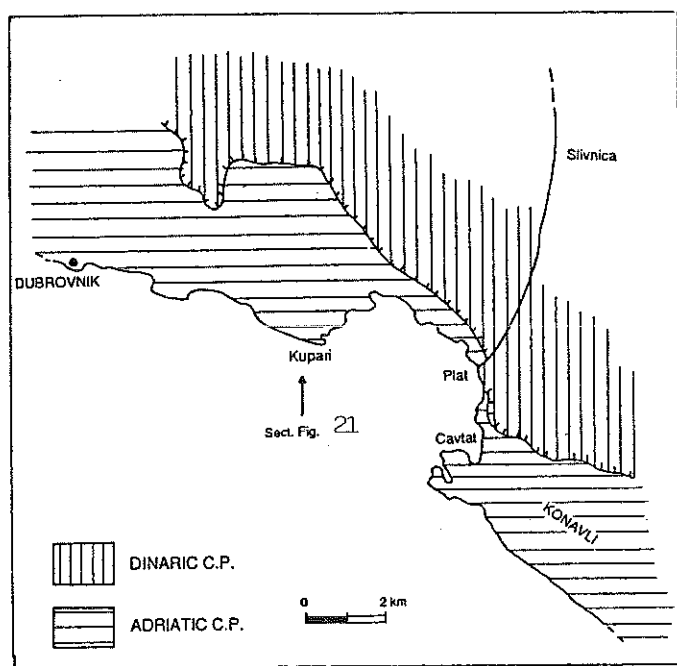


Figure 20. Schematic geologic map of the Dubrovnik-Konavli area.

Stop 1. Leaving the village of Kamenari; panoramic view of the Cukali-Budva sequences. Looking at the eastern coast of Boka Kotorska two different basinal sequences may be seen: the axial Lastva and the proximal Vrmac sequence (Figs. 24, 17A, 17B).

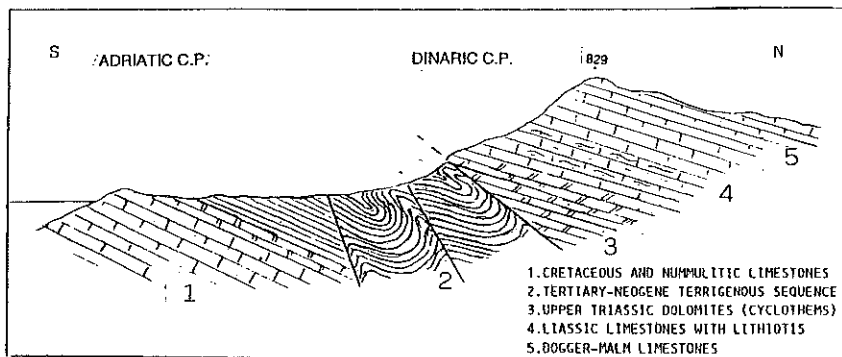


Figure 21. Schematic section of Dubrovnik-Konavli area showing the relationships between Dinaric and Adriatic Carbonate Platforms.

Stop 2. Verige - Vrmac sequence (Fig.17A). Along the roadcut a late Jurassic-early Cretaceous interval may be observed.

The Vrmac sequence is several hundred metres thick and includes Lower Ladinian to Lower Tertiary deposits. The lowermost outcropping sediments are Ladinian reef limestones with brachiopods, sphinctozoans, corals, hydrozoans, dasycladaceans and other algae, and calcareous breccias. From the Upper Ladinian to Cretaceous cherty limestones and more or less coarse breccias alternating to calcarenites and calcilutites follow, with intercalations of bedded chert and argillites. Some intervals, mostly Dogger and Lower Cretaceous in age, are highly silicified.

The Vrmac sequence is characterized by occurrence of Malm and Upper Cretaceous (mainly Senonian) massive breccias, resedimentated from shelf and scarp areas. The Vrmac sequence shows a strong lateral variability in lithology and thickness. About 5-6 km eastward of Boka Kotorska the sequence appears more dolomitized and somewhat less silicified with more coarse clastic material.

Stop 3. Perast, Boka Kotorska, Paleogene megabreccia (figs. 23, 24, and 15C).

Along the northern coast of the Boka Kotorska bay, from Morine towards Orahovac and Kotor, large bodies of megabreccias are commonly outcropping and form the southern flank of the surrounding mountain belt. Paleogene terrigenous deposits outcrop near Risan as well as in a narrow coastal belt from Orahovac to Kotor (Fig. 24).

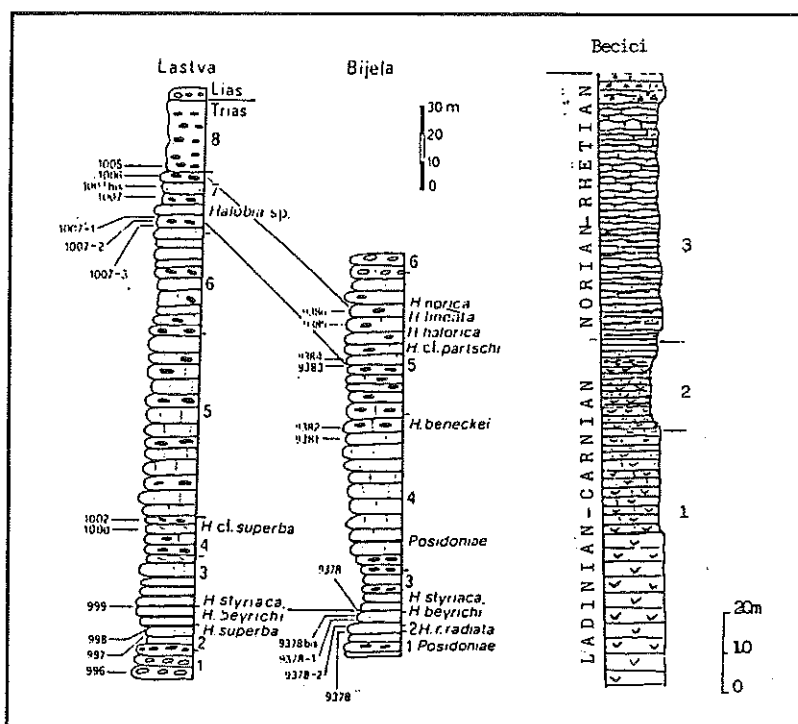


Figure 22. In the first two columns on the left: Upper Triassic sequences of the Cukali-Budva area (From CAFIERO & de CAPOA, 1980. In the column on the right: Triassic sequence at Becici (Porphyrit-Chert Formation type-area).

- 1: Pyroclastic, deposits; in the upper part green tuffs and jaspers with *Daonella indica* and *Daonella pichleri*
- 2: Alternating tuffs, claystones and limestones (ammonites, *Halobia*)
- 3: Jaspers, limestones and cherty limestones, partly nodular, with *Halobia* and radiolarians, grading into Liassic.

For a schematic map of the megabreccia distribution in the Montenegro-Split area see fig. 23. The Paleocene Lower-Middle Eocene megabreccias characterize the margin of the Dinaric Carbonate Platform towards the Cukali-Budva Basin and its northwestward prolongation. Concerning their geographic distribution and matrix it is possible to differentiate two breccia types: (a) breccia with barren greenish, commonly recrystallized matrix; (b) breccia with greenish fine grained calcareous matrix with planktonics or nummulitids.

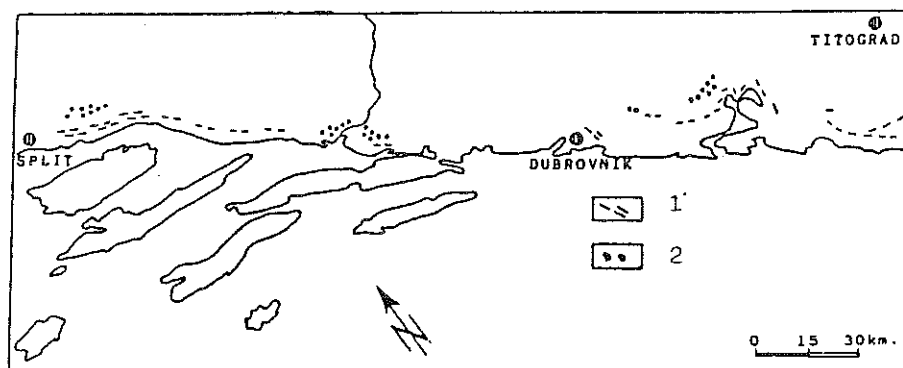


Figure 23 - Schematic map of the megabreccia distribution (From RADOICIC, 1969).
1 – breccia with microfossils in fine grained matrix; 2 – breccia with silt matrix.

The megabreccias overlie unconformably Cretaceous and Jurassic deposits (figs. 15-C, 15D, 26). Towards the basin they show lateral transition to proximal basin sequences. Their thickness is very variable ranging from hundred to several hundred meters; the lower interval (lower Paleocene), outcropping at Perast and Kotor, includes some levels of bioclastic limestones with rotalids, miliolids, rare planktonics and red algae debris.

Between Morine and Perast there are ruditic elements of different size including large blocks (3-4 m in size) supported by abundant matrix or by greenish marly limestones with planktonics pertaining to the *trinidensis-uncinata* zone and *pusilla pusilla* zone.

These megabreccias derived from the dismantling of a large part of the marginal belt of the Jurassic-Cretaceous Dinaric Platform, including some more internal sources. Even though they are well exposed, the megabreccias are commonly not reported on the geologic maps and until now they have not been studied.

After Perast (a nice small village, known for being home of famous sea captains) and going towards Kotor it is possible to see to the left mountain slopes formed by breccias and towards the right by the Cukali-Budva sequence (Vrmac).

After Kotor, along the road it is possible to observe Paleogene sequence (that exhibits red strata) pertaining to the Cukali-Budva domain (Vrmac sequence) in tectonic contact with the upper Cretaceous deposits of the same sequence.

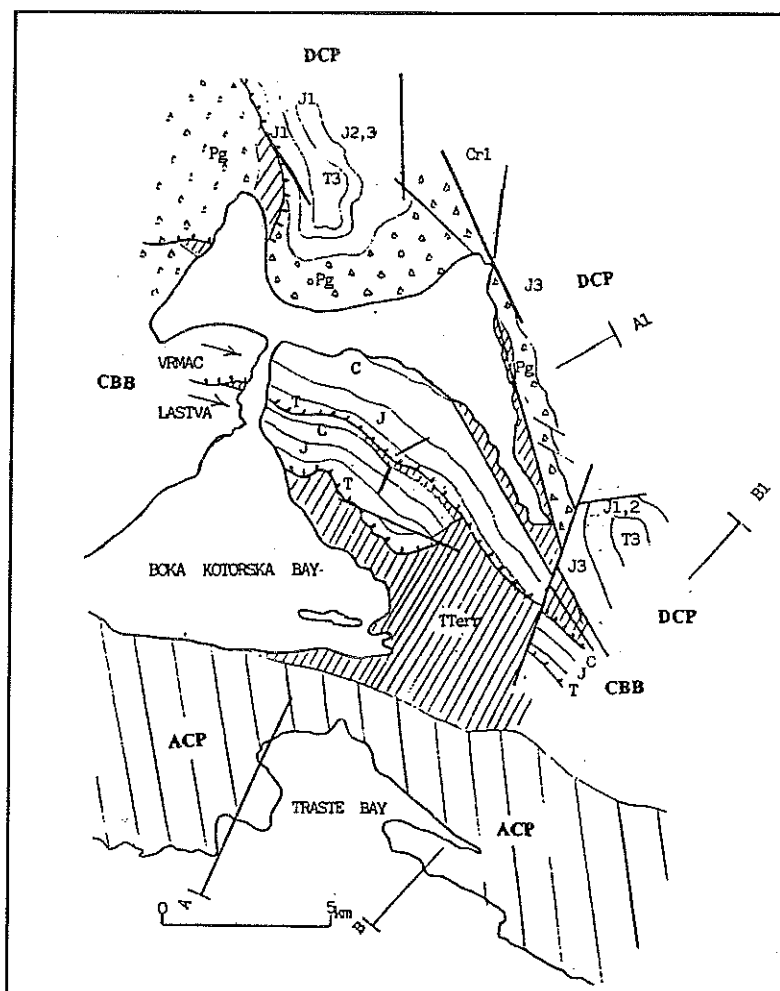


Figure 24. Geological map of Boka Kotorska area.

DCP – Dinaric Carbonate Platform; CBB – Cukali-Budva Basin; DCA – Adriatic Carbonate Platform. T – Triassic, J – Jurassic, C – Cretaceous, Pg – Paleogene breccias; Tt – Tertiary terrigenous sediments (undifferentiated).

Stop 4. On the slope of the Lovcen Mountain (about 24 km after Kotor) it is possible to get a panoramic view of the Boka-Kotorska bay and of its tectonic structure and to observe the Triassic-Lower Liassic sequence of marginal carbonate platform facies (figs 24, 25, 26).

a) Climbing from Kotor towards Njegusi, it is possible to see along the road and from quite elevated position the beautiful panorama of the Boka-Kotorska bay as well as the tectonic relationships of the different units (fig. 27). From South the panoramic view includes sequences from:

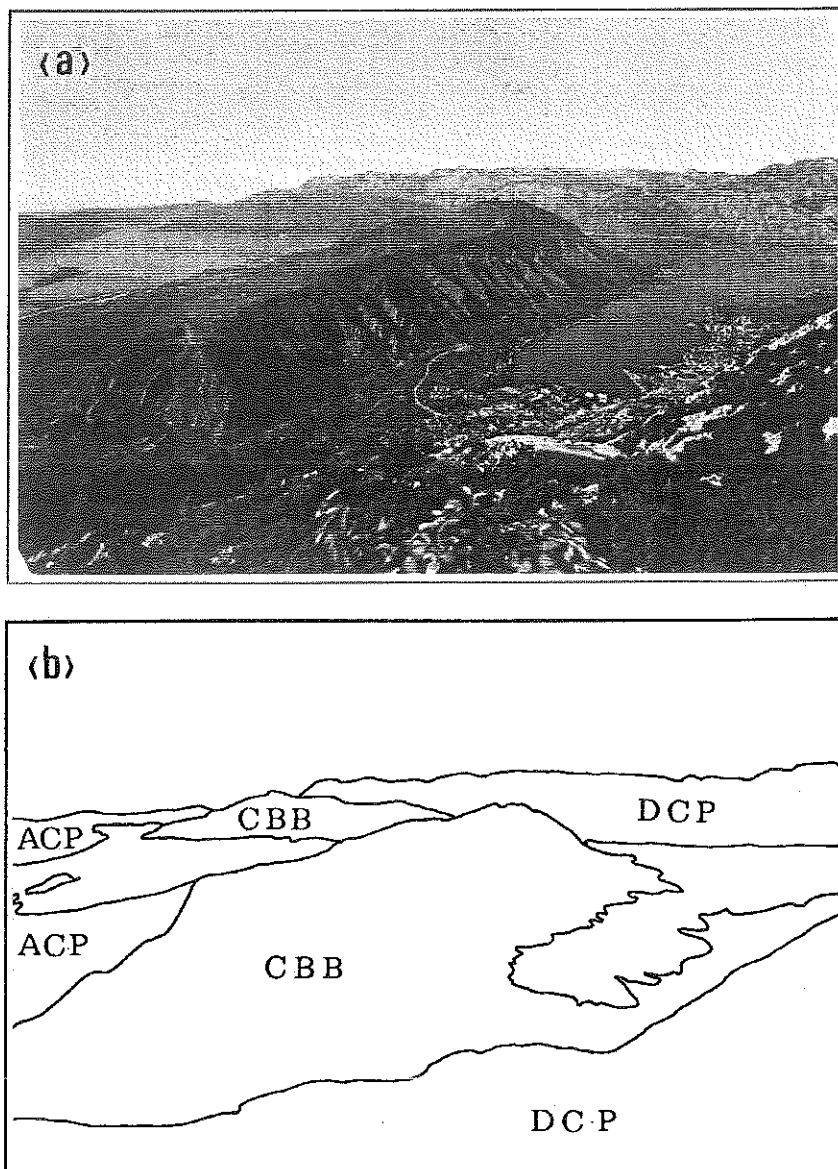


Figure 25 (a) Panoramic view of the Boka Kotorska bay. (b) ACP = Adriatic Carbonate Platform; DCP = Dinaric Carbonate Platform; CBB = Cukali-Budva Basin..

- the Adriatic Carbonate Platform represented by the Upper Cretaceous (Senonian) and Lower Paleogene carbonates and the Upper Eocene to Upper Miocene terrigenous sequences (de CAPOA et al., 1991, 1995);
- the Cukali-Budva basin, represented by the Lastva and Vrmac sequences (Trias-Upper Miocene, de CAPOA et al., 1994);

- the Dinaric Carbonate Platform, with Paleogene megabreccias and shallow water carbonates (Upper Triassic-Cretaceous).

b) Along the road, interesting outcrops may also be seen. The Lovcen Mountain pertains to the Southern margin of the Dinaric Carbonate Platform. On the original gentle slope of this platform during Liassic times periplatform deposits developed with well bedded, marly-nodular limestones with crinoids, brachiopods, *Lagenidae*, rare ammonites (mainly embryos). In other places reddish-brownish limestones with lagenids, *Involutina liassica* and rare ammonites also sedimented. Among the well known Pliensbachian brachiopod fauna collected in these deposits (CIRIC, 1949) may be mentioned: *Rhynchonella megabiensis*, *R. glycina*, *R. zugmayeri*, *R. amalthei*, *R. belemnitica*, *R. plicatissima*, *R. alfredi*, *R. gumbeli*, *R. variabilis*.

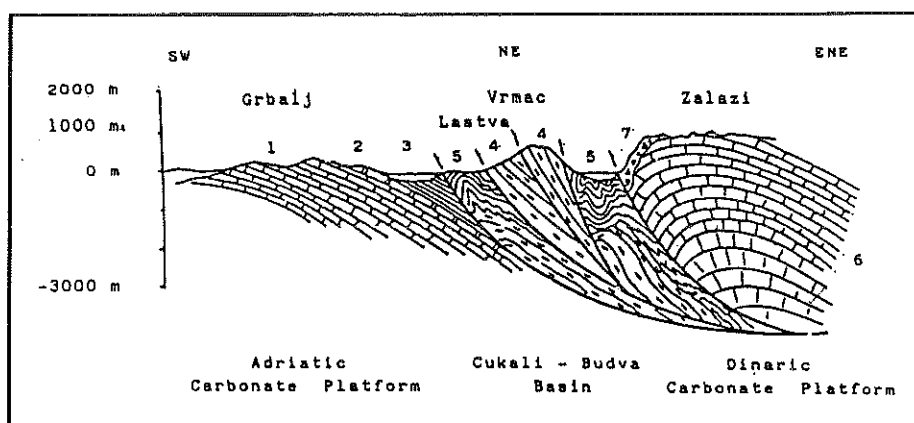


Figure 26. Geological section (A-A1) of the Boka Kotorska area (for location see Fig. 24). Adriatic Carbonate Platform: (1) Upper Cretaceous limestones and dolomites, (2) Middle Eocene limestones, (3) Upper Eocene to Upper Miocene terrigenous deposits. Cukali-Budva Basin: (4) Triassic to Upper Cretaceous and Paleogene clastics and pelagites, (5) Paleogene to Upper Miocene Pelagites and silicoclastics. Dinaric Carbonate Platform: (6) Triassic to Neocomian limestones, (7) Paleogene megabreccia.

The Lovcen area during Dogger-lower Malm time was characterized by the deposition of ooidal and ooidal-bioclastic limestones. In the upper Malm-earliest Neocomian a barrier-reef developed. It was characterized by the same reef building organisms, as in other Upper Jurassic reef of the Mediterranean realm (corals, hydrozoans, calcisponges, gastropods, diceratids, algae and foraminifers).

The following macrofossils have been recognized in the Lovcen-Boka reef: *Montlivaultia dilatata*, *M. subcompressa*, *M. greppini*, *Stylosmilia* aff. *corallina*, *Isastrea* cf. *foliacea*, *Calamophyllia flabellum*, *Cladophyllia* aff. *radiata*, *Rhytidogyra*, *Placogyra*, other corals, *Ellipsactinia polypora*, *E.*

ellipsoidea, *E. caprense*, *Sphaeractinia diceratina*, as well as *Heterodicerias luci*, *Pecten*, *Nerinea pseudobruntrutana*, *N. carpathica*, *N. tuberculosa*, *N. jeanjani*, *N. conoidea*, *N. acreon*, *Cossmannacea hohenegeri*, *Phaneroptyxis moreana*, *Trochalia pyramidalis*, *T. supramidalis*, *Endoplocus staszycii* etc. (KRKOVIC, 1967).

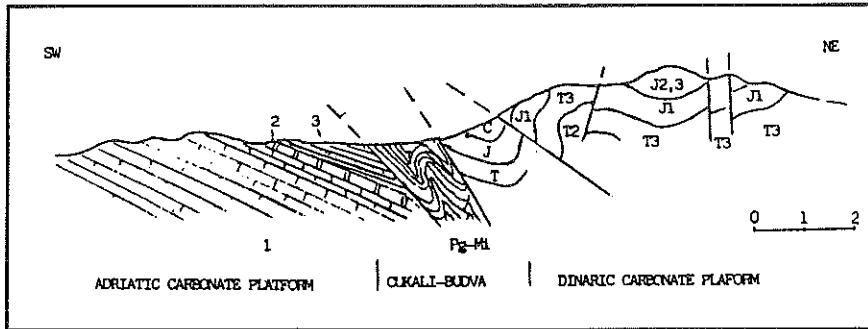


Figure 27. Geological section (B-B1) of the Boka Kotorska area (for location see fig. 24). Adriatic Carbonate Platform: (1) Upper Cretaceous and (2) Middle Eocene carbonates. Cukali-Budva Basin: Triassic, Jurassic and Cretaceous clastics and pelagites (T, J, C). Dinaric Carbonate Platform: Middle Triassic to Upper Jurassic limestones (T2; T3; J1; J2,3 respectively).

The Neocomian part of the reef complex (earliest Neocomian, Fig. 15C) is characterized by *Protopeneroplis trochangulata* and *Pseudocymopolia jurassica*.

Stop 5. Panoramic view of the Old Montenegro holokarst after Njegusi-Cetinje pass.

In this area the Liassic marginal belt, facing the adjacent Lovcen-Rumija basin, outcrops (fig. 4). The most common deposits are represented by dolomitized patch-reefs of Liassic age, with corals (*Thamnastrea*, *Isastrea*, *Epismilia*), sphinctozoans, molluscs, coriaceous, other algae and foraminifers. In particular, the sphinctozoan species *Stylothalamia columnaris* occur in a level near the Njegusi-Cetinje pass.

Northward lagoonal deposits of Liassic age bearing lithotids (interior platform) are widespread. The last visible range is represented by the Vojnik Mountain, formed by Upper Malm back-reef limestone pertaining the northern margin of the Dinaric Platform.

From Cekanje to Cetinje, in the outcrops may be seen good exposures of cyclothemic limestones and dolostones of upper Triassic age with *Megalodon triqueter dolomitica* (upper Carnian-Norian), *Megalodon complanatus* (Norian) and *Megalodon bockhi equivalvis* (Rhaetian) (KOCH, 1933).

In the village of Bjelosi, to the Southwest of Cetinje, type levels of *Chypeina besici* PANTIC (*Dasycladales*, Carnian) and *Turrispirillina minima* PANTIC (*Foraminifera*, Norian) occur.

After Cetinje, along the road for Budva, Triassic deposits of the Dinaric Platform outcrop; they are formed by middle Triassic dolostones and cyclothemic limestones, and by the Porphyrite-Chert Formation.

Leaving Brajici village towards Budva, tectonized sequences pertaining to the Cukali-Budva Basin, are widespread.

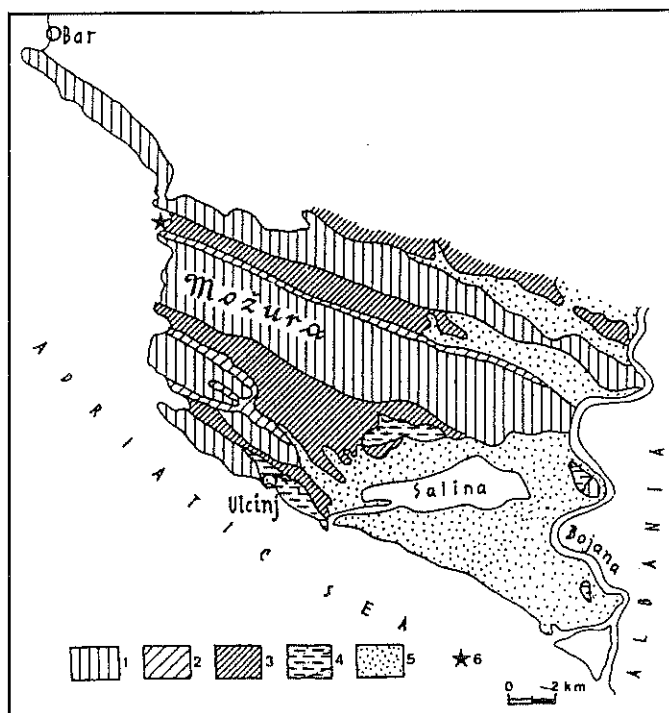


Figure 28. The area of Triassic "flysch" (outcrops in black) from M.N. DIMITRIJEVIC, 1967.

Stop 6 a, b. North of Kosljun, near Budva, (middle Triassic basinal facies).

(a) The outcrop of the stop 6a (a fresh roadcut) pertains to one of the "axial" sequences of the Cukali-Budva Basin. In this outcrop it is possible to observe:

- Anisian "flysch" (fig. 28), here characterized by a sequence of olistostromes with Paleozoic blocks and pebbles.
- Carbonatic interval, composed by resedimented carbonates (channellized conglomerates derived most probably from a lower Ladinian reef).
- Porphyrite-Chert Formation, in tectonic contact to the south with the overturned Jurassic -Cretaceous Kosljun series.

(b) Near Becici, along the road to Petrovac, outcrop pillow lavas pertaining to the Porphyrite-Chert Formation (first bridge to the left), then the type section of the Porphyrite-Chert Formation, follows (CIRIC, 1954; fig. 22).

Second day

Theme: "Mid Miocene" molasse (Messinian-Pliocene?); Cretaceous-Paleogene of the Adriatic Platform, relationships between Adriatic Carbonate Platform, Cukali-Budva basin and Dinaric Carbonate Platform; Anisian flysch; internal facies of the Dinaric Carbonate Platform.

Route: Sutomore-Ulcinj, Ulcinj-Niksic (km 199)

Stop 7. Ulcinj, "middle Miocene molasse" (fig 29, 30).

In the surroundings of Ulcinj, as well as in the town, terrigenous deposits overlap unconformably the Cretaceous and Eocene limestones and the "upper Eocene lowermost Oligocene flysch" (LUKOVIC & PETKOVIC, 1952). The maximum thickness of the "Miocenic" deposits reach 200 m. In this sequence, described by LUKOVIC & PETKOVIC are recognized from the bottom:

- gray-dark gray sands and sandstones,
- clayey sands and sandy clays with lenses of sandstones,
- "Leita limestone" (Lithothamnium limestone) that are characterized by a rich association of Rhodophyta, molluscs and foraminifers.

In this area well known outcrops named "Ulcinj locality" pertain to the middle and upper part of this sequence. Here VINASSA DE REGNY early this century collected a rich association of pelecypods: *Ostrea crassissima*, *O. germanitola*, *O. lamellosa*, *O. dogictalina*, *O. gigensis*, *Picnodonta cochlea*, *Cytherea multilamella*, *Chlamys multistriata*, *Aequipecten opercularis*, *A. scabrellus*, *A. malvinae*, *Pecten latissima*, *P. flabelliformis*, *P. gray*, *P. nicolai*, carbonates, 3. Late Eocene – Early Tortonian marly, clayes and arenaceous turbidites, 4. Messinian-Pliocene deposits, 5. Pleistocene, 6. North Mozura section. *P. revolutus*, *Radula martellii* etc.

Also a rich foraminifera association is present in these deposits, mentioned by TOMIC-DZODZO (1965) among which: *Spiroplectammina carinata*, *Marginulina costata*, *Elphidium flexuosum*, *E. crispum*, *Robulus cultratus*, *Orbulina universa*, *Eponides haidingeri*.

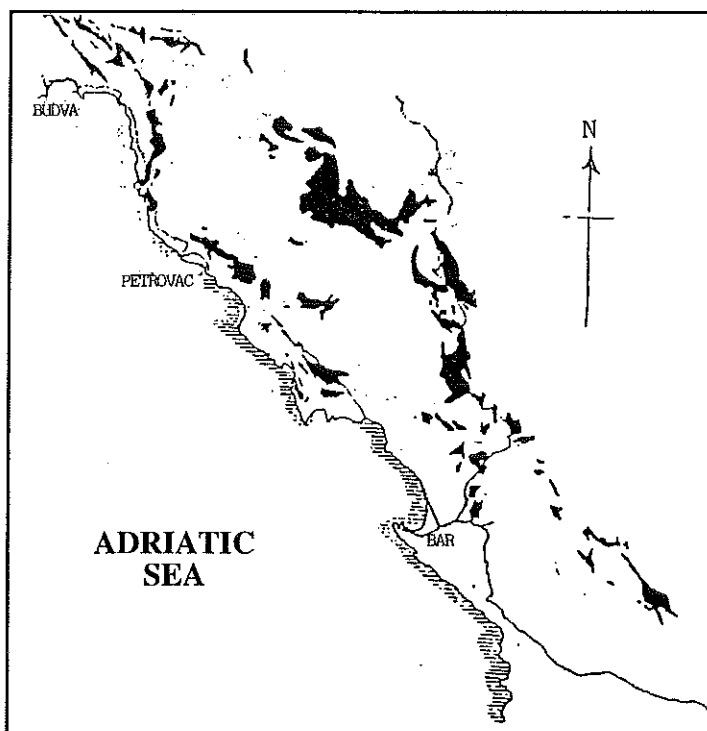


Figure 29. Schematic geological map of the Ulcinj-Bar area (after BURIC, 1966, modified in de CAPOA et al., 1995). 1. Late Cretaceous limestones, 2. Eocene

According to new biostratigraphic data (de CAPOA et al. 1991, 1995), these deposits are no older than Messinian. Note that some authors have reported a Pliocene age for this succession (V. DE REGNY, 1901; BAYLISS, 1978, fide RADOICIC, 1994).

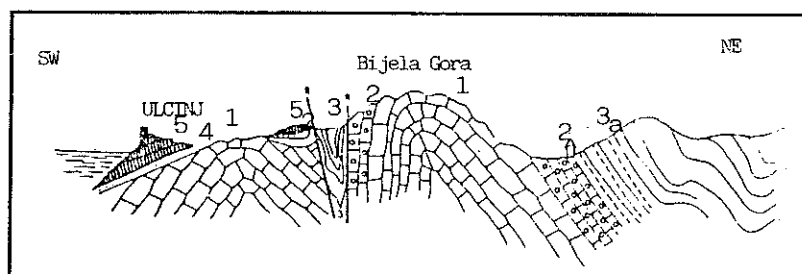


Figure 30. Geological section of the Ulcinj - Bijela Gora area (from LUKOVIC & PETKOVIC, 1952). 1. Upper Cretaceous rudistid limestones, 2. Eocene nummulitic limestones, 3. Eocene (to Oligocene?) flysch, 4. Middle Miocene fossiliferous sands, 5. Middle Miocene Lithotamnium limestones.

Stop 8 Ulcinj-Bar road; Adriatic Carbonate Platform

The Cretaceous and lower Paleogene carbonates of the Ulcinj-Bar develop upwards into terrigenous sequences known as "Upper Eocene-lowermost Oligocene flysch". These sequences are folded and thrust southward (LUKOVIC & PETKOVIC, 1952; VIDOVIC, 1958). From South to North, the following tectonic units may be singled out (fig 29):

- Ulcinj - Mendre
- Bijela Gora- Mavrijan
- Mozura - Brivska Gora and
- Volujica- Sasko Brdo

The Cretaceous deposits of the Ulcinj-Bar area are generally referred to the Senonian. Older sediments are documented only in subsurface (CANOVIC, 1965).

In the first two units a few tens of meters of dolomitic deposits seem to be the youngest member of the Cretaceous sequence and are formed by peritidal carbonates. The uppermost Cretaceous deposits are deeply karstified and their surface is filled with bauxites. Prebauxitic speleothems and *Microcodium* may be observed. The overlying carbonates are represented by Paleogene middle Eocene foraminiferal limestones with miliolids, *Spirolina*, *Coskinolina*, *Chrysalidina*, *Praerhyphidionina*.

The Cretaceous of Mozura anticline belt is well exposed along the new road Ulcinj-Bar. In this area minor faults are responsible for repetitive occurrences of this sequence. The oldest sediments are constituted by several tens of meters of breccias, followed at places by Early Senonian limestones or by Upper Turonian-Lower Senonian deposits. The middle part of Senonian (Uppermost Santonian and Campanian) is characterized by a rich association of rudists and foraminifers (*Katzeria*, *Bournonia*, *Gorjanovicia*, *Durania*, *Fundinia*, *Biradiolites*, and *Scandonea*, *Accordiella*, *Dicyclina*, *Minouxia*, *Keramosphaerina* etc.).

In the southern limb of the Mozura anticline dolomites represent the youngest member of the sequence, while northward limestones outcrop.

The uppermost part of the North Mozura Cretaceous sequence consists of limestones with rare foraminifers and rudist fragments, as well as of barren limestones, at places covered by bauxites.

The Paleogenic carbonates, about 50 m in thickness, consist of:

- Mudstone-wackestone with dasyclads and rare *Spirolina* or other foraminifers and foraminiferal packstone with *Nummulites* and *Fabiania casis*;
- Foraminiferal limestones (prevailing packstone) with *Discocyclinidae* and rare, small nummulitids (somewhere well sorted).

- Nodular limestones bearing small *Discocyclina* and planktonic forams, topped by a bryozoan rich hardground which passes into several meters of clayey-marly sediments and then into a thick sequence of terrigenous deposits.⁴

Toward Bar the road crosses the Volujica-Sasko Brdo belt, which is characterized by the same sediments as in Mozura. The breccias here reach a considerable thickness and result in a characteristic morphology. However, they are not reported on the geological maps of this area.

The Cretaceous breccias were known in the subsurface (wells Ulcinj 4 and Mozura 1, CANOVIC, 1965; ROKSANDIC & CANOVIC, 1971). There are not enough data about the stratigraphic position of these breccias. They are surely younger than lowermost Turonian and, in places, they reach the lowermost Senonian. Going towards Bar it is possible to see the Rumija Mountain (Dinaric Carbonate Platform, Cukali-Budva Basin and Adriatic Carbonatic Platform Domains).

The road from Bar to Virpazar crosses the southwestern side of the Rumija Mountain. Around the Sustasi and Tudjemili villages outcrops extensively the so-called Anisian "flysch". According to MILADINOVIC (1964) in this region are present also Werfenian sediments represented by arenaceous shales and micaceous-arenaceous marls bearing gypsum crystals and *Claraia claraia*, *Coelostylinia werfensis*, etc.

In the Sustasi village, the Anisian "flysch" includes olistolites of Permian age of different size and lithology (black shales, blackish and clayey limestones, reddish crinoidal limestones). From these blocks a well known fauna was collected, composed by brachiopods, corals, molluscs (KOCHANSKY-DEVIDE, 1951, 1965; KOSTIC-PODGORSKA, 1958).

Stop. 9. Tudjemili, panoramic view of the folded structure of the *Cukali-Budva Basin* sequences and Anisian "flysch".

At about 2 km after the Sustasi village, the road crosses part of the basinal (proximal) Jurassic sequence, composed by partly silicized, ooidal calcarenites.

A road cut at Tudjemili village shows the distal facies of the "Anisian flysch", which in some levels shows brachiopods in life position.

⁴ To prepare the guidebook for this excursion, a study of calcareous nannoplankton in this well exposed "Eocene flysch" section along the northern limb of the Mozura anticline was performed. Preliminary data related to the lower part of the succession indicated an age between Late Eocene (NP24 biozone Martini, 1971) and Late Miocene (NN3 biozone Martini, 1971).

The results stimulated further biostratigraphic investigations which has given evidence of a Middle-Upper Miocene age for the upper part of this succession. These biostratigraphic results have had an important implication for the deformation timing of the External Dinarides that, contrary to common opinions, has to be rejuvenated from Early Tertiary to the end of Neogene, (RADOICIC et al, 1989; de CAPOA et al. 1991, 1995), at least as far as their Southern sectors are concerned.

Between Rumija and Sozina Mountains, in the Crmnica area, along a new road, middle Triassic and Jurassic deposits outcrop (mainly resedimented ooidal calcarenites) that are also present in the northern side of Sozina Mountain.

The road between Limljani and Boljevicci crosses Anisian conglomerates, upper Anisian ammonitic limestones (*Han Bulog* facies) and Ladinian pillow lavas.

Stop 10. Limljani-Boljevicci, (a) panoramic view of West Crmnica structures and (b) Anisian conglomerates and pillow lavas (Porphyrite-Chert Formation).

The panoramic view of the West Crmnica shows:

- Southward, on the left, the Sozina-Rasovatac chain (a complex tectonized area), where outcrop marginal sequences of the Dinaric Platform belt and of its slope toward basal sectors, with megabreccia (distinguishable for its karstified relief). The latter is in tectonic contact with a Triassic tectonized sequence made of Werfenian to Upper Triassic dolomites of the Dinaric Platform; part of the above information comes from subsurface data based on two wells near Brijegi and Brceli villages (ROKSANDIC and CANOVIC, 1970).

In the Sotonici-Limljani area the upper part of the Anisian "flysch" sequence is characterized by calcirudites, ooidal calcarenites and marls with flute casts; the middle part of this sequence shows conglomeratic intercalation.

In the Sotonici-Bujan-Limljani sector Anisian "flysch" conglomerates (150 m thick) outcrop. Here, M.N. DIMITRIJEVIC (1967) distinguishes:

- lower part represented by angular elements and poorly cemented pebbles,
- middle part characterized by pebbles, graded at places and
- upper part well cemented and graded, with calcarenites and matrix rich sandstones, more frequent upwards.

The conglomerate elements range, generally, from 5 to 20 cm in size (but may at times reach 100 cm).

According to DIMITRIJEVIC (1967, pp 228 and 230-232) two types of pebbles are present: the 41% represented by elements of oldest rocks and 59% of the same age of the sequence; a microconglomeratic and arenitic matrix is also present.

In the Anisian "flysch" rich fossil association has been found; it is represented by brachiopods (*Spiriferina*, *Mentzelia*, *Rhynchonella*), crinoids (*Encrinurus*), bivalves (*Myophoria*), ammonites (*Ceratites*, *Arcestes*) and a very rich algal (dasycladaceans *Macroporella*, *Physoporella*, *Oligoporella*) and foraminifera (*Meandrospira*, *Glomospira*, *Glomospirella*) (BUKOVSKI 1904, BESIC 1959, PETKOVIC & MIHAJLOVIC 1935, PANTIC 1967).

Near Sotonici village, in the Anisian "flysch" at Glibovi, there is a type locality of *Pilamina densa* PANTIC.

The Boljevici village area is one of the most famous localities of the *Han Bulog* type Ammonitico Rosso. From this locality come the very rich fauna (69 species of ammonites) collected and described by MARTELLI (1904).

After Boljevici, along the road it is possible to see Upper Triassic dolostones.

Stop 11. Vranjina Island, panoramic view of Skutari Lake and Rumija Mountain.

The road between Virpazar and Vranjina crosses upper Triassic carbonates along the Skutari Lake.

The Rumija Mountain visible in the background pertains to the southern Dinaric Platform margin. After a first carbonate platform development (Upper Triassic-lowermost Liassic) the Liassic margin shifted toward the interior areas of the platform where marginal facies with patch-reefs developed, gradually passing southward to the Lovcen-Rumija slope to basin facies and to Cukali-Budva Basin (fig.4).

During the Dogger, this margin shifted basinward and in Lower Malm a barrier-reef developed (colonial corals, *Ellipsactinia* and other hydrozoans, diceratids, gastropods, algae and forams).

KRKOVIĆ (1965) set up, from this reef association, one new genus *Krainastrea* and 11 new species of corals:

Aplosmilium rotunda, *Aplosmilium grandicosta*, *Rhypidogyra meandra*, *Phytogyra costata*, *Phytogyra densasepta*, *Comophylia ostros*, *Comophylia dichotoma*, *Dermoseris granulum*, *Epistreptophyllum ellipsoides*, *Dermosmilium labrata* and *Krainastrea murici*.

During the middle-upper Malm the reef facies shifted basinward and in this zone lagoonal facies develop (*Charophyta* limestones, *Paraurgonina* limestones, *Clypeina jurassica* limestones etc.).

The Vranjina Island is formed by Upper Triassic-lowermost Liassic cyclothems and its upper part by Lower-Middle Liassic deposits.

Moving Northwards (Dodosi, fig. 31), over the above strata typical facies of grey and dark-grey limestones with *Lithiotidae*, other bivalves and foraminifera outcrop (*Durga crassa*, *Durga nicolissi*, *Durga trigonalis*, *Gervilleia buchi*, *Pachymegalodus angustus*, *Placunopsis ghelensis* and *Orbitopsella praecursor* etc.).

Leaving the Skutari lake and its marsh shores, near the Moraca River, we enter the Zeta plain, South of Podgorica (Titograd). To the left it is possible to see the Western Zeta hills (fig. 31) constituted by internal platform sequence (Dogger-Turonian and until Senonian Northward).

Close to Grbavci and Lekici villages there is the type locality of *Salpingoporella dinarica* RADOVIĆ (Aptian limestone).

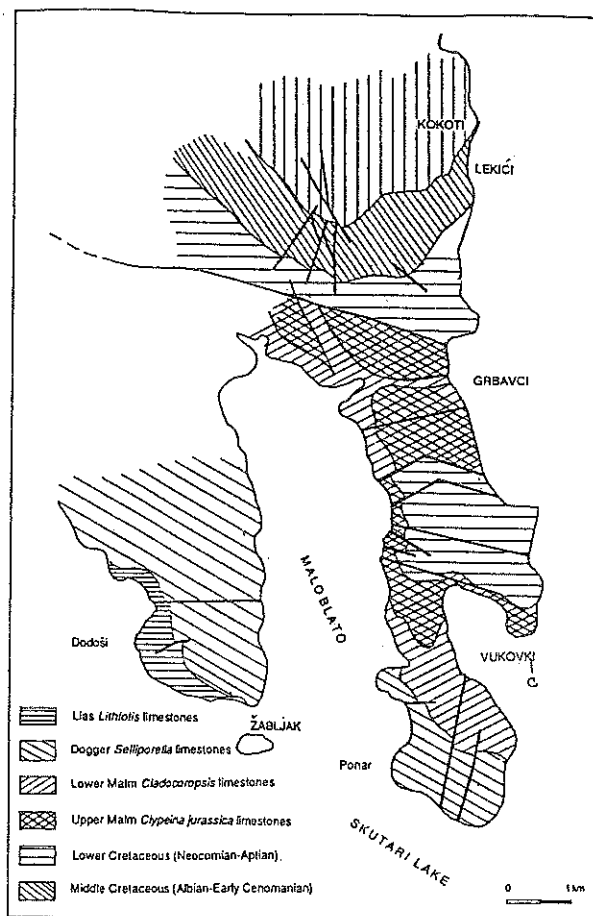


Figure 31. Geologic map of the western edge of the Zeta plain (from RADOICIC, 1959).

Stop 12. Podgorica (Titograd)suburbs; panoramic view, to East and North-northeast of the mountains surrounding the Zeta-Djemovsko Polje.

At Fundina, E and NNE of Zeta-Djemovsko Polje (figs. 32,33) it can be seen the Decic anticline, where a sequence of Kimmeridgian to Lower Miocene age outcrops (de CAPOA & RADOICIC, 1994).

In the Zeta region also internal platform facies (Upper Jurassic to Cretaceous) outcrop.

- *Cladocoropsis* limestones (Malm)
- Charophyta and *Salpingoporella grudii* limestone (Malm)
- *Clypeina jurassica* limestones and dolomitized limestones (Malm), "aberrant *Tintinnidae*" limestones and dolomitized limestones (lowermost Neocomian),

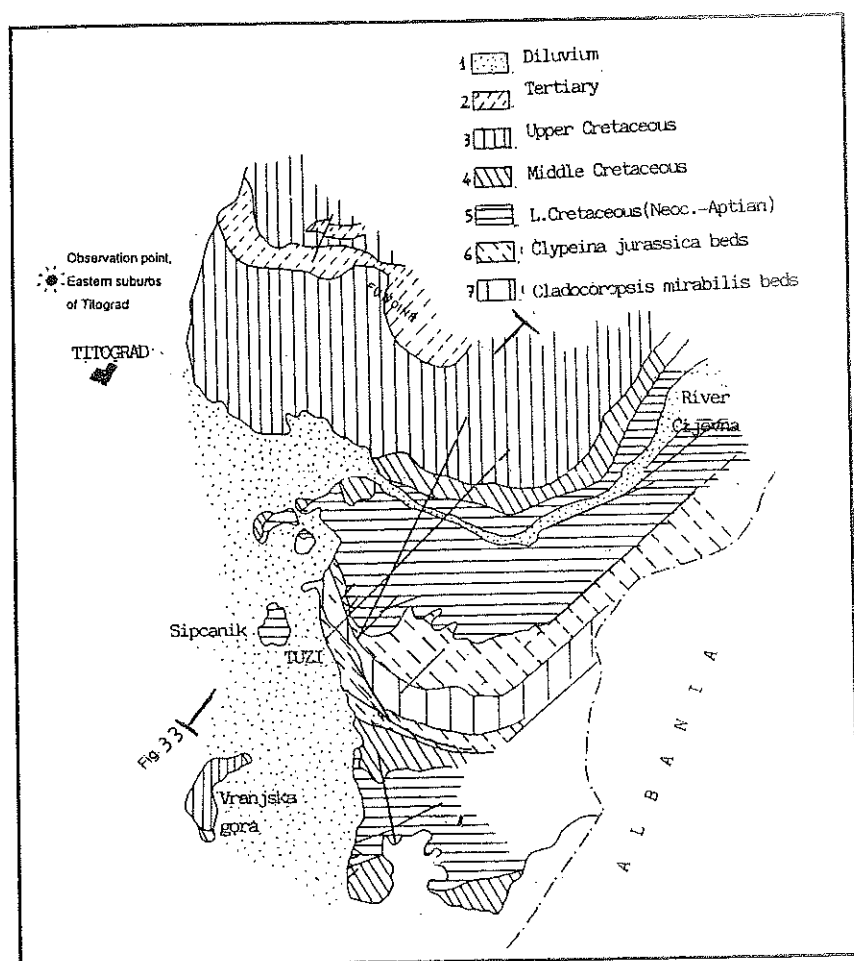


Figure 32. Geological map of the northern edge of the Zeta plain (from RADOICIC, 1959).

- Limestones and dolomitized limestones with ostracods, coprolithes and dasyclads (*Triploporella neocomiensis*, *Humiella*, *Clypeina solkani*, *Salpingoporella annulata*, *Salpingoporella* spp.), (Neocomian),
- Limestones and dolomitized limestones with dasyclads and forams (*Actinoporella podolica*, *Salpingoporella muhlbergi*, *Salpingoporella cemi*, *Salpingoporella melitae*, *Salpingoporella dinarica*, than miliolids, orbitolinids, *Debarina*, *Sabaudia*, *Nezzazata* and other), (Barremian and Aptian),
- Dark blackish limestones and dolomitized limestones with ostracods, rare foraminifers and *Glaucina* (Albian),
- Limestones and dolomitized limestones with (a) *Chondrodonta* and other ostreids (including bituminous shaly beds), (b) *Neoiraquia convexa*,

(c) *Nummuloculina*, *Biconcava*, *Nezazzata*, *Broeckinella*, (d) *Radiolitidae*, other pachyodonta, gastropods etc. (Cenomanian),

- Limestones and dolomitized limestones with *Aeliosaccus kotori* and rare foraminifers, with rudistids, gastropods, *Pseudocyclammina sphaeroidea* etc. (Turonian-lower Senonian).

During Late Senonian the Zeta furrow spread and rudistid reef complex developed in this area (back-reef calcarenites, orbitoidal-coral-rudistid grainstone, floatstone and bindstone).

In the Mt. Decic area there are the type localities of dasyclads *Salpingoporella grudii* (RADOICIC), *Salpingoporella cemi* (RADOICIC) and *Triploporella neocomiensis* RADOICIC.

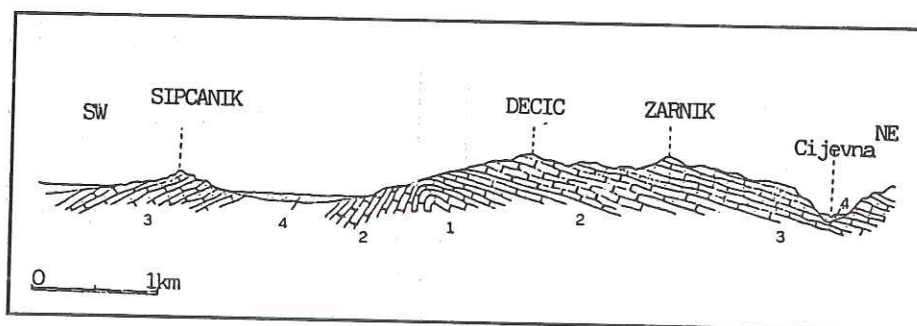


Figure 33. Geological section across the northern edge of the Zeta plain (from RADOICIC, 1958, for location see Fig. 32). 1. Malm: *Cladocoropsis* and *Chypeina jurassica* limestones and dolomites; 2. Lower Cretaceous: dolomites and limestones, 3. Quaternary: alluvial deposits.

Stop 13. Podgorica (Titograd)-Niksic road, panoramic view from Bogetici village to Zeta River valley, where the Zeta intraplatform furrow sequences outcrop (Figs. 4, 13 C and D).

In the Gornja Zeta River valley outcrop the Zeta furrow sediments:

1) Senonian limestone with rudistids and orbitoids, passing to *Globotruncana cherty* limestone and to

2) *Globotruncana* calcareous marls and Paleogene to Miocene sequence (known as "limestone and mary flysch" and "Eocene flysch", PAVIC, 1970).

Fig. 34 shows part of Zeta valley structure to the north of Zeta River.

Third Day

Theme: Marginal (Vojnik, Gacko) and internal (Niksic, Trebinje) area of the Dinaric Platform

Route: Niksic-Vojnik area, Vojnik-Trebinje-Tjentiste (Km 250).

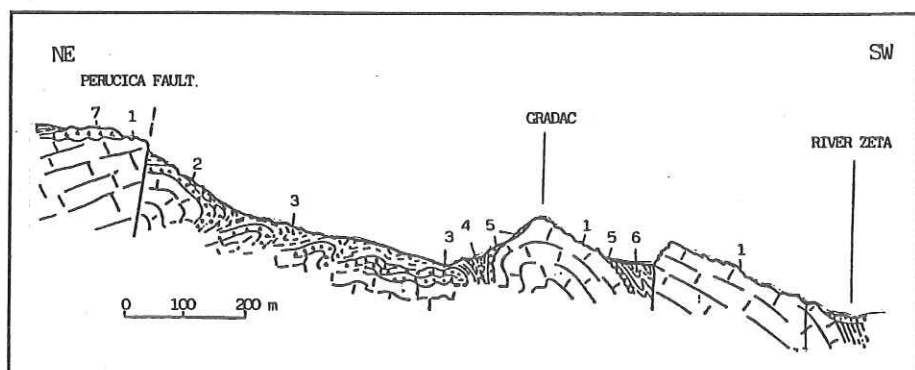


Figure 34. Geological section across Zeta intraplateform furrow (from PAVIC, 1970). 1. Limestones with rudists, 2. Breccia with Orbitoids, 3. Green and red marls with *Globotruncana*, 4. Green and red marls with breccia intercalations (with *Discocyclus*), 5. Coarse breccia, Gradac conglomerates, 6. Paleocene, flysch, 7. Upper Tertiary? Breccia with nummulites.

Stop. 14. Lipova Varan, Ammonitico Rosso succession (Trias-Dogger)

Leaving Niksic (20 km to the North), and passing through Gornje Polje, the zone of Vojnik Mountain is reached. This region is well known for its "Ammonitico Rosso" outcrops of Late Liassic age. The "Ammonitico Rosso" fauna includes: *Hildoceras bifrons*, *H. comensis*, *Harpoceras fluitans*, *H. cf. bicarinatus*, *H. cf. discoides*, *Erycites perczely*, *Lithoceras francisci*, *Lithoceras francisci compressa*, *L. velifer*, *L. fimbriatum*, *Phylloceras gardanum*, *Ph. heterophyllum*, *Cenoceras striatum*, *C. truncatus*, *Gramnoceras thouarensis*, *Juraphyllites limatus*, *Calliphylloceras nilssoni*, *Nautilus terebratus*, *N. samseyi*, *Atractites beticus*, *A. inglatus* and *Involutina liassica*, and a large lagenid population, etc. (BESIC, 1948; RADOICIC, 1956, 1964; MIRKOVIC, 1965).

Object of this stop is an outcrop along the road that leads at the Lipova Ravan village, characterized by rich Ammonitic levels.

The succession can be followed from the last layers of Late Triassic age (*Triasina hantkeni*) until Dogger, and then into Lower Cretaceous. A short hiatus in the Lower Lias probably occurs. Also in the eastern area of Vojnik (Praga, Lokva Milankovac), a hiatus marked by bauxites occurs between the Liassic "Ammonitico Rosso" and the lagoonal limestones with *Clypeina jurassica* (Upper Malm, PAVIC, 1956). In this area the gap corresponds to the Bajocian and lowermost Malm. Along the marginal area, over the Lower Malm reef complexes developed lagoonal environments characterized by limestones and dolomitized limestones with *Clypeina jurassica*.

During Upper Malm, reefs prograded basinwards. The paleoenvironmental evolution during Jurassic and Neocomian times in Vojnik region, like in the all northern margin of the Dinaric carbonate platform, is similar to that of Rumija.

Finally it must be noted that to the North of Vojnik, in Komarnica-Duzi canyon, the reef of Barremian-Aptian age with corals, calcareous sponges, hydrozoans, caprinids, requienids, dasyclads and foraminifera, is well preserved.

Stop 15. Gornja Polje, Vir, Triassic bauxite, Raiblian Beds.

Coming back to Niksic and stopping at Gornja Polje village (Vir locality) there are bauxite deposits interposed between Middle Triassic and Carnian (Raiblian) strata (PANTIC-PRODANOVIC & RADOSEVIC, 1978).

The following sequence may be seen:

- Anisian limestone with *Macroporella alpina*, *Oligoporella* sp., *Tubiphytes obscurus*, *Endothyra salaji*, *E. wirzi*, *Duostomina alta*, *Meandrospira* sp., and others,

- Ladinian limestone with radiolarians and echinoderms in the lower part, and *Tubiphytes obscurus* in the upper part.

- Bauxites,

- Raiblian beds with *Cuspidaria gladius*, *Avicula kokeni*, *Avicula* cf. *caudata*, *Trigonodus problematicus*, *Trigonodus carniolicus* and *Pecten incognitus* (PANTIC, 1956).

Also the Carnian levels of Niksic area are generally fossiliferous; among the others are present: *Temnotropis carinata*, *Pseudoscalites paucistriatus*, *Purperioidea carithiformis*, *Loxotomella hoernesii*, *Gonodon mellingi*, *Megalodon hoernesii*, *Megalodon rimosus*, *Pleurotomaria (Worthenia) exsul* (PANTIC, 1956; BURIC, 1966).

From Vir we come back to Niksic, crossing the central part of the Dinaric carbonate platform.

The Zeta furrow strata outcrop in the Niksic area and NW of Niksicko Polje, as a thin level below the Kuci imbricate.

Western Montenegro is a region particularly rich in bauxite of Triassic, Jurassic and Cretaceous age. In regards to the bauxite age, it is worth noting that in the geological literature of Montenegro 8 bauxite layers are reported, between Middle Trias and Upper Cretaceous (PAVIC, 1953, 1956, VLAHOVIC, 1964, BESIC et al., 1965).

This conclusion is based:

- a) On the age of their substratum

- b) In some cases on the incorrect stratigraphic data.

Fig. 11 shows stratigraphic columns with Jurassic and Cretaceous red and white bauxites of Western Montenegro.

The road to Trebinje, crossing the bauxitiferous region of Kuside, Trubjela and Padbozur, often meets well exposed layers with white bauxite intercalated

between Lower or lowermost Neocomian and Cenomanian. Close to Viluse village, a road cut shows Liassic sediments with brachiopods, lagenids, crinoids and rare ammonites (Fig. 4).

Stop 16. Klobuk, panoramic view on the anticline of Grahovo-Lastva (Triassic-Jurassic) (Figs. 35 and 36)

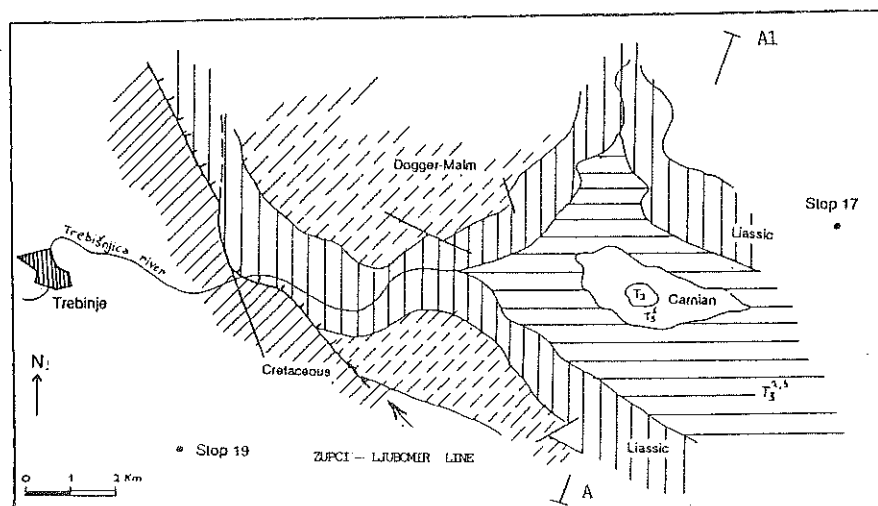


Figure 35. Geological map of the Grahovo-Lastva anticline.

The Jazina River cuts the anticline core where Ladinian and Carnian rocks outcrop (cherty limestones, limestones, shaly and marly limestones, thin bedded limestones, marls and coal) with *Cuspidaria gladius*, *Myophoria kefersteini*, *Pseudoscalites armatus* etc. (BUKOWSKI, 1901) and fish remains (*Ptycholepis* sp., ANDJELKOVIC, 1976). Upwards follow Upper Triassic-Lowermost Liassic dolomites, then the typical *Lithiotis* facies and the interior platform sequence (Dogger-Lower Cretaceous) with bauxite of Malmian age.

Stop 17. Jazina (Lastva). Short look at the Carnian rocks along the northern limb of the anticline (Fig. 36).

From the Jazina (Lastva) to Trebinje, along the Trebisnjica valley, a Liassic outcrop containing *Lithiotidae* may be followed for about 5 km, after a bridge. Then the road crosses a thrust (of Liassic rocks above Cenomanian strata) on the left side of the river.

Stop 18. Jasen, panoramic view on the Trebisnjica valley; with the thrust of Zupci - Ljubomir (Fig. 35).

The thrust line is evident on the orographic left side, where liassic verticalized beds with *Lithiotidae* are tectonically superposed on the Upper Cretaceous rocks. The thrust plane is also displaced by subsequent faults.

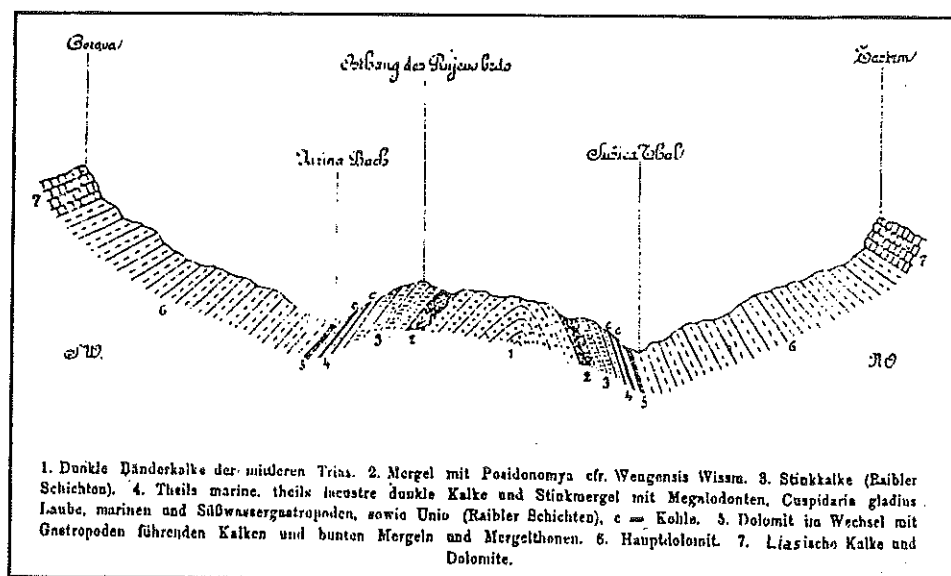


Figure 36. Geological section of the Lastva anticline (form BUKOWSKY, 1901); for location see A-A1 in Fig. 35.

Moving towards Bileca, Jurassic and Cretaceous outcrops pertaining to the Dinaric internal platform domain may be seen. On the right the very gentle morphology marks outcrops of Portlandian-Neocomian dolomites. Close to the Bileca (now below the water level of an artificial lake) is the spring of the Trebisnjica river.

Between Bileca and Stepen, Middle and Upper Cretaceous outcrops are crossed. At Stepen we come in Zeta furrow area, where Eocene carbonates and Miocene terrigenous deposits outcrop.

From Stepen it is possible to see on the left, the large Baba Mountain fold, that is part of the Bjelasnica antichine. Here the sequence is represented by Upper Malm levels with *Chelypeina jurassica* and limestones with fibrous-radial ooids of lowermost Neocomian age.

In the Stepen-Gacko area, the Zeta furrow joins with the Durmitor Basin (Fig. 4).

After Stepen the road crosses the Gatacko Polje, where an intramountain molassic basin with coal occurrences of Neogenic age is preserved. The Gatacko Polje Neogenic basin is 15 km long, 4 km wide and filled by more than 300 m of sediments (MUFTIC, 1964), the main coal level is found at a depth between 12 and 25 m. The sequence is referred to the Upper Miocenian (MILOJEVIC, 1966) on the basis of the mollusc association (*Fossarulus*, *Enchilus*, *Limneus*, *Orygoceras*, *Helix* etc.). The flora is mainly represented by *Glyptostrobus europaeus*, *Taxodium distichum mioceanum*, *Taxodium dubium*, *Pterinospermum vagans* (MUFTIC, 1964).

As the deformation time in External Dinarides is post-Tortonian (de CAPOA &, 1995), the age of this intramountain molassic basin must be no older than the Pliocene.

Stop 19. Miholjaca-Gacko section (fig. 13A). Domerian with ammonites.

Crossing Gatacko Polje area, from Avtovac the road to Tjentiste and Foca after 1 km, leads to the black shaly limestones with ammonites of Domerian age (GAKOVIC, 1986), pertaining to the adjacent Zalomka-Gacko basin domain.

This large basin associated with northern platform margin was directly connected with the intraplatform Zeta furrow (Fig. 4). Part of this Liassic basin is characterized by euxinic sedimentation. The Liassic limestones, mostly calcilutites with or without chert intercalations, evolve into Dogger and Lower Malm limestones to which carbonate clastics follow. The marginal complex of the Dinaric Platform during the Upper Malm prograded basinward over most of these adjacent basinal strata. However, in some areas the basinal sedimentation continued during the Malm, reaching the Aptian-Albian.

Stop 20. 2 km northward. Upper part of the same section.

In the same road, scarp breccia and reefs deposits follow to the Dogger-Lower Malm sequence. Without any apparent interruption, the Upper Malm scarp breccias pass towards into an Upper Cretaceous polygenic breccia (fig. 13A).

Upward follows a Senonian-Middle Eocene sequence represented by alternating breccias, microbreccias, calcarenites and calcilutites, calcareous marls and conglomerates (turbidite sequence of the South Durmitor basin).

Fourth Day

Theme: North Durmitor- South Durmitor tectonic relationships; Dinaric ("High Karst") thrust.

Route: Tjentiste-Dubrovnic-Split (km 365)

In the Tjentiste area Werfenian sediments outcrop at the base of the Triassic carbonates of the Volujak and Zelengora Mts in which the Sutjeska river is cut. After Tjentiste coming back to Gacko, Middle Triassic dolomites, foraminiferal

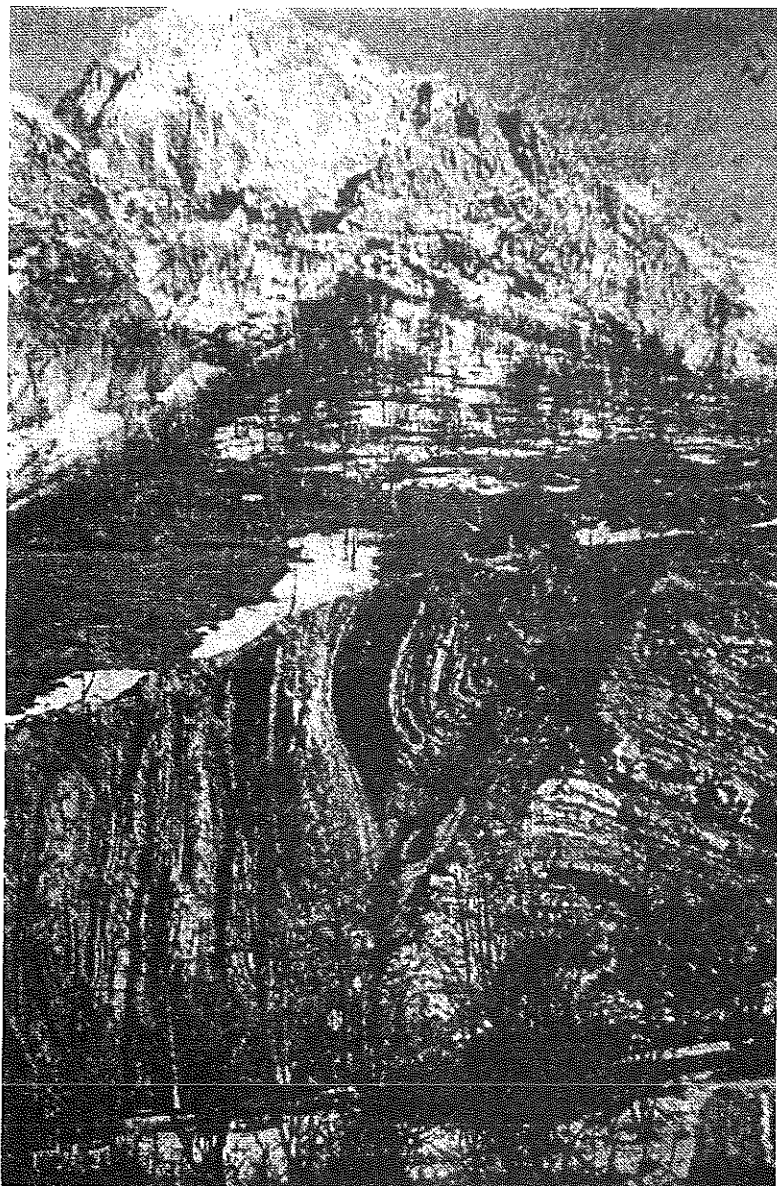


Figure 37. North Durmitor Carbonate Platform rocks thrust on the folded strata of South Durmitor Basin (from MOJICEVIC, 1974).

limestones, breccias and Porphyrite-Chert Formation and Upper Triassic limestones are crossed (generally occurring at the mountain tops).

In the Suha area the valley is characterized by huge blocks fallen from steep scarps and made by Carnian reddish limestones.

Leaving Suha, close to a tunnel, the road crosses deformed Strata (Lias to Malm carbonates, and Tithonian-Neocomian flysch outcropping in a tectonic window) which are in tectonic contact with Triassic rocks.

The amount of deformation of the Volujak and Zelengora Mountain increase close to the overthrust of North Durmitor Platform on the South Durmitor basinal unit (Fig. 37).

Stop 21. Sutjeska valley, panoramic view toward the North Durmitor overthrust (Triassic--Jurassic) on the South Durmitor basin, (Tithonian-Neocomian "flysch")

The road crosses the Tithonian-Neocomian flysch sequence until Cemerno; this sequence, even if folded, shows great thickness.

Stop 22. Cemerno, Tithonian-Neocomian "flysch" and panoramic view toward the Lebrsnik thrust above South Durmitor.

The Lebrsnik Upper Cretaceous clastic carbonates overthrust the Cretaceous-Eocene clastic folded sequence, pertaining to the same basin, already crossed after Stop 20.

Stop 23. Ivanica (Croatia), panoramic view toward Dinaric (High Karst) thrust (Figs. 20 and 21).

From Ivanica, through Bileca and Trebinje, return to Dubrovnik (and to Split in the original tour).

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Acknowledgements. The present paper was prepared as an introduction to a field trip across the External Dinarides, led by the Authors during September 1988, on the occasion of the 74th Congress of the Geological Society of Italy and as an extension of a geologic field trip across the Southern Apennines. The Southern sector of the Dinarides is one of the most interesting and spectacular regions of the Periadriatic mountain belt. The reason to select this area for a field trip was the similarity between the Southern Apennines and the Southern Dinarides, which appear like a mirror image of each other, as far as the Mesozoic - Tertiary sedimentary evolution and the general tectonic organization are concerned; even though the Dinarides do not show the late extensional imprint which was superimposed to the Southern Apennines as a consequence of the Tyrrhenian basin development. The guidebook was never published.

A few years ago several colleagues suggested that we publish this guidebook or at least its geologic introduction considering the lack of information available on the geology of the region.

After a couple of years that the manuscript was literally lost in the drawers of the Geologic Society of Italy (where it was submitted for publication in early 1997, and never rejected or accepted), we have decided to re-organize the manuscript and to publish it in the "Rendiconto dell'Accademia delle Scienze Fisiche e Matematiche" of the Società Nazionale di Scienze, Lettere e Arti in Napoli.

For lack of time and economic reasons we have retained the sketchy form the figures had in the original guidebook, because it was not possible to re-draw all of them; we ask to be forgiven for that.

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**Xylocopini (Hymenoptera: Apidae: Xylocopinae)
presenti nelle collezioni entomologiche italiane:
il Museo Zoologico dell'Università di Napoli Federico II.**

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Presentato dal socio Giuseppe Geraci
(Adunanza del 4/12/1999)

Key words: Xylocopini, *Xylocopa*, *Lestis*, *Proxylocopa*, Revision, A. Costa Collection, Naples Zoological Museum, new names, Italy.

Abstract - The Xylocopini specimens contained in the entomological collections of Zoological Museum of Naples are 84, belonging to Achille Costa, Vincenzo Ragazzi, Mario Salfi and Ferdinando Solari collections. The recognized species are as follows: *Lestis areatus*, *L. bombylans*, *Proxylocopa olivieri*, *Xylocopa aestuans*, *X. amauroptera*, *X. amedei*, *X. aruana*, *X. augusti*, *X. auripennis*, *X. barbata*, *X. brasilianorum*, *X. caerulea*, *X. caffra*, *X. ciliata*, *X. circumvolans*, *X. coronata*, *X. dejanii*, *X. fimbriata*, *X. frontalis*, *X. frontalis* var. *morio*, *X. inconstans*, *X. iris*, *X. latipes*, *X. loripes*, *X. nautlana*, *X. nigrita*, *X. nigrita* var. *nigripars*, *X. olivacea*, *X. oudemansi*, *X. perforator*, *X. perversa*, *X. similis*, *X. tabaniformis*, *X. valga*, *X. violacea*, *X. virginica*, *X. sp.* Three new Xylocopini names are reported in the Museum catalogue (specimens lost): *X. abdominale*, Natal (Southafrica); *X. flavilabra* Piemonte (Italy); *X. sticticum*, Sicily (Italy).

Riassunto - Gli Xylocopini del Museo Zoologico di Napoli, appartenenti alle collezioni Achille Costa, Vincenzo Ragazzi, Mario Salfi e Ferdinando Solari, contano 84 esemplari appartenenti alle seguenti specie: *Lestis areatus*, *L. bombylans*, *Proxylocopa olivieri*, *Xylocopa aestuans*, *X. amauroptera*, *X. amedei*, *X. aruana*, *X. augusti*, *X. auripennis*, *X. barbata*, *X. brasilianorum*, *X. caerulea*, *X. caffra*, *X. ciliata*, *X. circumvolans*, *X. coronata*, *X. dejanii*, *X. fimbriata*, *X. frontalis*, *X. frontalis* var. *morio*, *X. inconstans*, *X. iris*, *X. latipes*, *X. loripes*, *X. nautlana*, *X. nigrita*, *X. nigrita* var. *nigripars*, *X. olivacea*, *X. oudemansi*, *X. perforator*, *X. perversa*, *X. similis*, *X. tabaniformis*, *X. valga*, *X. violacea*, *X. virginica*, *X. sp.* Nei cataloghi del Museo sono presenti tre nomi nuovi per gli Xylocopini (esemplari perduti): *X. abdominale* (Natal, Sudafrica); *X. flavilabra* (Piemonte, Italia); *X. sticticum* (Sicilia, Italia).

1 - INTRODUZIONE

Gli Xylocopini sono una delle numerose tribù costituenti la famiglia Apidae (*sensu* Roig-Alsina & Michener, 1993) e sono suddivisi in tre generi: *Xylocopa* Latreille, 1802 (cosmopolita), suddiviso in 47 subgeneri; *Lestis* Lepeletier & Serville, 1828 (australiano); *Proxylocopa* Hedicke, 1938 (Paleartico-subcontinente indiano) suddiviso in 2 subgeneri (Hurd & Moure, 1963; Eardley, 1987, 1993; Vicidomini, 1997b, 1997c). Sono comunemente denominate "api carpentiere grandi" (in contrapposizione alle api carpentiere nane = Ceratinini) (Vicidomini, 1995) in quanto scavano il loro nido in substrati vegetali, ottenendo tunnel che poi

verranno parzialmente riempiti con celle pedotrofiche (per una rassegna dell'etologia della nidificazione vedi: Vicidomini, 1996b, 1996c, 1997a); fanno eccezione le specie del genere *Proxylocopa* le quali sono passate secondariamente a nidificare nel sottosuolo (Vicidomini, 1997a). Il presente contributo fa parte di un progetto di studio che ha lo scopo di costituire un "data base" delle collezioni entomologiche italiane, contenenti specie di Xylocopini (e.g.: Vicidomini, 1996d, 1997d, 1997e, 1997f; Vicidomini & Priore, 1999); suo oggetto specifico sono gli Xylocopini presenti nel Museo Zoologico dell'Università di Napoli Federico II all'interno del quale viene custodita la collezione "Achille Costa", una delle più importanti collezioni storiche entomologiche d'Italia (Maio et al., 1995; Poggi & Conci, 1996). Viene inoltre realizzata una breve scheda biometrica per i vari esemplari e vengono inoltre pubblicati tutti i dati relativi gli Xylocopini registrati nel Catalogo del Real Museo Zoologico (AA.VV., 1845-1900) (dati in Tab. 2) e nella letteratura prodotta da A. Costa nel periodo 1862-1871 e D. Guiglia (Guiglia, 1953) (dati in Tab. 3). Non è stato possibile invece consultare l'inventario amministrativo, andato perduto nel dopoguerra. Il materiale viene poi completato fornendo il quadro sinottico della seconda collezione entomologica dell'Università Federico II, quella custodita presso l'Istituto di Entomologia e Zoologia Agraria "Filippo Silvestri" di Portici.

2 - METODI

2.1 - Revisione

Gli esemplari delle collezioni entomologiche del Museo Zoologico sono stati visionati riportando tutti i dati presenti sui cartellini originali, nonché quelli desunti dalla consultazione del Catalogo (AA.VV., 1845-1900). Ogni esemplare a revisione ultimata è stato dotato di due cartellini, senza sostituire gli originali, i cui dati compaiono anche nel testo: il primo riporta specie, autore, numero e dati della cassetta in cui è custodito, rigo all'interno della cassetta in cifre romane, posizione all'interno del rigo in cifre arabe; il secondo cartellino riporta invece, tutti i dati presenti sui cartellini originali ed in più quelli desunti dal catalogo (e.g.: luogo e data di cattura, eventuale collezione di appartenenza, *legit*); viene poi indicato, tra parentesi quadre nel testo, anno di catalogazione, volume e n.° progressivo; il n.° dell'inventario amministrativo (n.° inv.), in inchiostro rosso sul Catalogo, e le eventuali note, vengono riportati tra parentesi tonde; le note aggiuntive personali vengono poi di seguito indicate; la bibliografia relativa al singolo esemplare viene citata tra parentesi tonde nel testo. Viene inoltre riportata, tra parentesi subito dopo il sesso dell'esemplare, anche l'eventuale erronea determinazione di A. Costa. La revisione degli esemplari è stata effettuata secondo i seguenti autori: Smith (1874) in, Vicidomini (1997c); Maa (1938, 1968, 1970); Hurd & Moure (1963); Hurd (1978); Eardley (1983, 1993) e confrontando alcuni esemplari con quelli contenuti nelle collezioni del Museo Civico di Storia Naturale "Giacomo Doria" di

Genova dotato di numerosi esemplari tipici (e.g.: Maa, 1968). Alcune specie vengono indicate come *complex* in quanto il loro status sistematico è ancora irrisolto, celando probabilmente serie di specie gemelle ancora da descrivere (e.g.: Hurd, 1978).

2.2 - Collocazione

Gli esemplari sono contenuti in cinque cassette, con fondo a torba, appartenenti a tre collezioni differenti: Collezione dell'Italia Meridionale (cassetta *Apidae: Xylocopa, Bombus*. n.° 64: 71x200x260 mm); Collezione Extra-Europea (cassette 60x195x262 mm: *Apidae: Xylocopa*; n.° 339. - *Apidae: Xylocopa*; n.° 360. - *Apidae: Lestis, Eulaema*; n.° 341); Collezione del Massiccio del Pollino (1 cassetta senza numero; 60x190x260 mm; vedi Guiglia, 1953).

2.3 - Scheda Biometrica

I caratteri biometrici misurati per ogni esemplare sono di seguito elencati e vengono riassunti in Tab. 1 (cfr.: Eardley, 1983): lunghezza totale, misurata dall'apice anteriore del capo fino all'apice posteriore del metasoma = LT; lunghezza dell'ala anteriore, misurata dalla tegula (inclusa) fino all'apice = LAA; lunghezza dell'ala posteriore, misurata dalla sua base fino all'apice = LAP; lunghezza del metasoma, misurato tra i due estremi dell'asse antero-posteriore del tagmata = LM; larghezza del capo, misurata come massima distanza tra le genae = LgC. Tali misure sono state rilevate con calibro ed approssimate a 0.05 mm. Oltre alle misure sono state annotate anche le condizioni, i segmenti e le parti mancanti, citandole con le seguenti sigle: S solo spillo e cartellino (= esemplare perduto); C capo; A antenne; F flagello; O occhi composti scavati da antreni; M mesosoma lacerato da antreni; P protorace; Z zampe; B basitarso, T tarsomeri; A, M, P, rispettivamente, anteriore, mediano, posteriore; (+) destra; (-) sinistra; ECD esemplare contorto e deformato. Nella scheda biometrica ogni esemplare sarà indicato col proprio numero di collocazione nella cassetta originale (riga e posizione all'interno della stessa). Nel testo con (S.c.) si indicano poi quegli esemplari infestati dall'acaro foretico *Sennertia* (*Sennertia*) *cerambycina* (Scopoli, 1763) (Chaetodactylidae) (Vicidomini, 1996a).

3 - MATERIALE

Genere *Lestis* Lepeletier & Serville, 1828

Lestis bombylans (Fabricius, 1775): 1F, Australia, [1845-1872; I; 8568], (Costa, 1864a), (341 I I). — 1F, Australia, [1845-1872; I; 8568], (341 I I 2).

Lestis areatus Smith, 1851: 1M, Sudaustralia, Adelaide, [1845-1872; I; 8569; n.° 63], (Costa, 1866), (341 I 3).

Genere *Proxylocopa* Hedicke, 1930

Proxylocopa (Proxylocopa) olivieri (Lepeletier, 1841): 1F, (*X. fuscata*), Persia, [1887-1888; III; 39215], (dono del direttore), (64 II 5).

Genere *Xylocopa* Latreille, 1802

Subgenere *Afroxylocopa* Hurd & Moure, 1963

Xylocopa nigrata (Fabricius, 1775): 1M, Spagna, Cadiz, Cadice, [1845-1872; I; 8555], (Costa, 1866), (339 I 3).

Xylocopa nigrata (Fabricius, 1775) var. *nigrata* (Leveque, 1933): 1F, Sierra Leone, [1889-1890; III; 40885], (n.° inv. 37020; acquisto Standinger), (339 III 3).

Xylocopa nigrata (Fabricius, 1775) var. *nigripars* (Leveque, 1933): 1F, Spagna, Cadiz, Cadice, [1845-1872; I; 8555], (Costa, 1866), (339 IV 1).

Xylocopa oudemansi (Leveque, 1933): 1F, Africa Occidentale, Haitifi, (339 II 4).

Subgenere *Alloxylocopa* Maa, 1939

Xylocopa circumvolans Smith, 1873: 1M, (*X. kalinowsky*), Giappone, Iukoman, [1887-1888; III; 39216], (dono direttore), (360 VI 3). — 1F, (*X. kalinowsky*), Giappone, Iukoman, [1887-1888; III; 39216], (dono direttore), (360 VI 4).

Subgenere *Biluna* Maa, 1938

Xylocopa auripennis Lepeletier, 1841: 1F, [1845-1872; I; 8554], (339 III 1).

Subgenere *Copoxyla* Maa, 1954

Xylocopa iris (Christ, 1791): 1F, (*X. minuta* var. *amethystina*), Italia, Calabria, (64 I 3). — 1M, S. Sev., (64 I 5). — 1M, (64 I 6). — 1F, Italia, Campania, Torre Annunziata Centrale (NA), Coll. V. Ragazzi, 14 VI 1916, (64 II 2). — 1M, (*X. cyaneescens*), Italia, Sicilia, Monti Madonie (PA), (64 II 6). — 1M, Italia, Sicilia, Monti Madonie (PA), (64 III 2). — 1M, (*X. minuta*), Italia, Sardegna, [1885; III; 36183], (n.° inv. 22373; viaggio del direttore; vedi, Costa, 1886), (64 III 3). — 1F, Italia, Campania, Torre Annunziata Centrale (NA), Coll. V. Ragazzi, 24 V 1914, (64 III 4).

Subgenere *Cyaneoderes* Ashmead, 1899

Xylocopa caerulea (Fabricius, 1804): 1F, Giava, [1845-1872; I; 8561], (Costa, 1864a), (360 I 4).

Subgenere *Koptortosoma* Gribodo, 1894

Xylocopa aestuans Linnè, 1758: 1F, Egitto, Cairo, [1873; II; 16507], (dono Pr. Panceri in seguito ad un suo viaggio in Egitto; vedi: Costa, 1875; Panceri & Gasco, 1876), (64 II 3). — 1M, Egitto, Cairo, [1873; II; 16507], (dono Pr. Panceri in seguito ad un suo viaggio in Egitto; vedi, Costa, 1875), (64 III 1). — 1F, Abissinia, [1898-99; III; 49007], (360 VI 1).

Xylocopa amauroptera Perez, 1901: 1F, [n.° 62], (360 II 5).

Xylocopa aruana Ritsema, 1876: 1F, Indonesia, Isola Aru, [1876; II; 23114], (cambio), (360 V 6).

Xylocopa caffra (Linné, 1767): 1F, Sudafrica, (360 III 2).

Xylocopa coronata Smith, 1860: 1F, Nuova Guinea, (360 II 1).

Xylocopa olivacea Spinola, 1838: 1F, India, (360 III 5). — 1M, India, (360 III 6).

Xylocopa perversa Wiedemann, 1824: 1F, (*X. mesoxantha*), Giava, [1875; II; 21939], (dono del direttore fatto al ritorno dal suo viaggio per diverse città d'Europa), (360 V 2). — 1F, (*X. mesoxantha*), [1890-1891; III; 41634], (n.° inv. 37876; acquisto Frohstorfer; discaricato), (360 V 4). — 1F, (*X. mesoxantha*), [1890-1891; III; 41923], (n.° inv. 38146; acquisto Frohstorfer), (360 V 5).

Xylocopa sp. (*aestuans-caffra* clx): 1M, Hnozona C.W. (località non leggibile ma certamente africana), [1845-1872; I; 8565], (360 IV 3).

Xylocopa sp.: 1F, (*X. zonata*), Giava, [1845-1872; I; 8562], (Costa, 1871), (360 II 4).

Subgenere *Megaxylocopa* Hurd & Moure, 1963

Xylocopa fimbriata Fabricius, 1804: 1F, D. Bolivia, 10 IV 1904, (64 I 2).

Xylocopa frontalis (Olivier, 1879): 1F, Brasile, [1845-1872; I; 8558], (Costa, 1866), (360 I 1). — 1F, Brasile, [1890-1891; III; 41747], (cambio Saussure), (360 II 2).

Xylocopa frontalis var. *morio* (Olivier, 1879): 1M, (*X. cubaecola*), Cuba, [1845-1872; I; 8557], (339 V 4).

Xylocopa nautilana Cockerell, 1904: 1F, (*X. latipes*), Indie (località probabilmente errata od incompleta; specie dell'America Centrale), [1845-1872; I; 8549], (339 I 2).

Subgenere *Nanoxylocopa* Hurd & Moure, 1963

Xylocopa ciliata Burmeister, 1876: 1F, Uruguai, [1876; II; 23115], (cambio), (339 II 2). — 1F, Uruguay, [1876; II; 23115], (cambio), (360 V 1).

Subgenere *Neoxylocopa* Michener, 1954

Xylocopa augusti Lepelletier, 1871: 1M, Uruguay, Monte Video; G. Sichel leg., [1845-1872; I; 8556], (esemplare donato dal Dr. G. Sichel di Parigi; vedi, Costa, 1864b), (339 V 1). — 1F, Uruguay, Montevideo; G. Sichel leg., [1845-1872; I; 8556], (esemplare donato dal Dr. G. Sichel di Parigi; vedi, Costa, 1864b), (339 V 2). — 1M, Uruguay, Montevideo; G. Sichel leg., [1845-1872; I; 8556], (esemplare donato dal Dr. G. Sichel di Parigi; vedi, Costa, 1864b), (339 V 3).

Xylocopa brasilianorum clx (Linné, 1767): 1F, (*X. aeneipennis*), U.S.A., California, [1890-1891; III; 41746], (cambio Saussure), (360 II 3). — 1M, (*X. morio*), U.S.A., Sud California, [1890-1891; III; 41744], (cambio Saussure), (360 IV 6).

Xylocopa similis Smith, 1874: 1F, Colombia, Bogotá (Deyr.), [1845-1872; I; 8564], (Costa, 1866), (360 III 1).

Subgenere *Notoxylocopa* Hurd, 1956

Xylocopa tabaniformis Smith, 1874: 1M, Messico, [1845-1872; I; 8567], (Costa, 1864a), (360 IV 1). — 1M, Messico, [1845-1872; I; 8567], (Costa, 1864a), (360 IV 2). — 1F, Messico, [1845-1872; I; 8567], (Costa, 1864a), (360 V 3).

Subgenere *Platynopoda* Westwood, 1840

Xylocopa latipes (Drury, 1773): 1M, Australia, Queensland, Capo York, [1876; II; 22310], (339 I 1). — 1M, Giava, [1890-1891; III; 41633], (n.° inv. 37875; acquisto Frohstorfer; discaricato), (339 II 1).

Xylocopa perforator Smith, 1861: 1M, (*X. latipes*), India, [1845-1872; I; 8553], (339 III 2).

Subgenere *Rhysoxylocopa* Hurd & Moure, 1963

Xylocopa amedei Lepeletier, 1841: 1F, Russia, [1895-96; III; 45046], (n.° inv. 41430; acquisto Standinger; discaricato), (64 II 4).

Subgenere *Shoenherria* Lepeletier, 1841

Xylocopa barbata (Fabbricius, 1775): 1F, (*X. muscaria*), Brasile, [1845-1872; I; 8551], (Costa, 1862), (339 I 4).

Xylocopa loripes Smith, 1874: 1M, America, [1845-1872; I; 8560; n.° 62], (360 I 5). — 1M, America, [1845-1872; I; 8560], (360 VI 2).

Subgenere *Xenoxylocopa* Hurd & Moure, 1963

Xylocopa inconstans Smith, 1874: 1M, (*X. nigrita*), Sierra Leone, [1889-1890; III; 40885], (n.° inv. 37020; acquisto Standinger), (339 II 3).

Subgenere *Xylocopa* Latreille, 1802

Xylocopa valga Gerstaecker, 1872: 1M, Italia, Campania, Torre Annunziata Centrale (NA), Coll. V. Ragazzi, VI 1912, (64 I 7). — 1F, Italia, Abruzzo, Maiella, (64 IV 1). — 1F, Italia, Sicilia, Monti Madonie (PA), (64 IV 2). — 1F, Italia, Campania, Postiglione, Fiume Calore (SA), Coll. M. Salfi, V-VI 1953, (64 IV 3). — 1F, (64 IV 4). — 1F, Italia, Calabria, Gioia Tauro (RC), [1892-1893; III; 43121], (n.° inv. 39379; acquisto Standinger), (64 IV 5). — 1F, (*X. violacea* Guiglia det.), Italia, Piemonte, Voltaggio, Appenino genovese (AL), Coll. F. Solari, 8 IX 1951, (64 V 2). — 1F, Italia, Campania, Postiglione, Fiume Calore (SA), Coll. M. Salfi, V-VI 1953, (64 V 3).

Xylocopa violacea (Linnè, 1758): 1E, Sud Italia, [1845-1872; I; 8548], (Costa, 1862) (64 I 1). — 1M, Italia, Campania, Castellamare di Stabbia (NA), Coll. V. Ragazzi, IX 1912, (64 II 1). — 1F, Italia, Sicilia, Monti Madonie (PA), (64 V 1). — 1F, Portogallo, (64 V 4). — 1M, Italia, Campania, Castellamare di Stabbia (NA), Coll. V. Ragazzi, 26 V 1914, (64 V 5). — 1M, Italia, Campania, Castel Volturno (CE), (64 VI 1). — 1M, Italia, Campania, Postiglione, Fiume Calore (SA), Coll. M. Salfi, V-VI 1953, (64 VI 2). — 1F, (339 IV 2). — 1F, Italia, Calabria, Colloredo (CS), 900-1100 m, La Greca & Sarà leg., Guiglia Det., 16 II 1950, (Guiglia, 1953), (Pollino I 1). — 1F, Italia, Calabria, Vallone Santicelli (CS), Massiccio Pollino, Ruffo Leg., VI 1950, (Guiglia, 1953), (Pollino I 2).

Subgenere *Xylocopoides* Michener, 1954

Xylocopa virginica (Linnè, 1771): 1F, (*X. virginica*), U.S.A., Tennessee, [1890-1891; III; 41745], (cambio Saussure), (360 III 3). — 1M, (*X. virginica*) U.S.A., New York, (360 III 4).

Subgenere *Zonohirsuta* Maa, 1938

Xylocopa dejanii Lepeletier, 1841: 1F, (*X. collaris*), India, [1845-1872; I; 8559], (Costa, 1864a, 1866), (360 I 2). — 1F, (*X. collaris*), Ceylon (= Sri Lanka),

[1890-1891; III; 41922], (n.° inv. 38145; acquisto Frohstorfer), (360 I 3). — 1M, Cambogia, [1845-1872; I; 8566], (Costa, 1866), (360 IV 4). — 1F, (*X. collaris*), India, [1845-1872; I; 8559], (Cfr.: Costa, 1864a, 1866), (360 IV 5). — 1M, Giava, [1892-93; III; 42677], (n.° inv. 38866; acquisto Frohstorfer; scaricato), (360 VI 5). — 1M, Giava, [1892-93; III; 42678], (n.° inv. 38867; acquisto Frohstorfer; scaricato; *praegd. var.*), (360 VI 6).

4 - DISCUSSIONI & CONCLUSIONI

Nelle cassette sono presenti 84 esemplari appartenenti alla tribù Xylocopini oltre a 2 Apoidea non Xylocopini (posizione 64 I 4 = *Bombus* sp.; 360 V 7 = gen. indet.), i quali rientrano in 37 specie di cui 1 con il solo subgenere determinato (posizione 360 II 4); questo esemplare viene determinato da A. Costa, sia sul cartellino, sia nel Catalogo, che in Costa (1871), come *X. zonata* Klug; tale nome è oggi attribuito ad una specie di *Proxyllocopa* ma data la notevole differenza morfostrutturale tra l'esemplare in collezione (subgenere *Koptortosoma*) ed il genere *Proxyllocopa*, è da escludere la loro conspecificità; inoltre *P. zonata* è stata istituita come specie solo nel 1930 (Alfken, 1930). Dato ulteriormente interessante è che non è nota alcuna specie di Xylocopini dell'Indonesia le cui caratteristiche siano simili a quelle dell'esemplare 360 II 4 (Maa, 1939; Lieftink, 1955, 1956, 1957); sono inoltre stati consultati i testi classici dell'entomologia del 1700-1800 (Fabricius, 1775, *Systema Entomologiae*. Fabricius, 1793, *Entomologia Systematica emendata et aucta*. Latreille, 1802, *Histoire naturelle, générale et particulière des crustacés et insectes*. Latreille, 1809, *Genera Crustaceorum et Insectorum*. Dalla Torre, 1896, *Catalogus Hymenopterorum*), tra i quali gli stessi articoli di Klug (1807, 1808) contenevano informazioni sugli Xylocopini, ma non è stato osservato alcun riferimento per *X. zonata*. Lo status tassonomico dell'esemplare 360 II 4, che rimane quindi irrisolto, esula comunque gli scopi di tale contributo e sarà oggetto di un futuro studio. Sono presenti tutti e 3 i generi costituenti la tribù. *Lestis* è presente con ambedue le specie note e riconosciute come valide (Hurd & Moure, 1963; Houston, 1992), costituendo ciò un caso unico tra tutti gli istituti italiani (Vicidomini, in preparazione). *Proxyllocopa* è presente con 1 specie, dato rilevante in quanto il Museo Zoologico di Napoli fa parte dei pochissimi istituti italiani in cui sono conservati esemplari appartenenti a questo genere: Istituto di Entomologia di Portici; Musei di Genova, Torino, Verona (Vicidomini, in preparazione). *Xyllocopa* è presente con 34 specie appartenenti a 17 subgeneri (36% dei subgeneri conosciuti) (vedi: Hurd & Moure, 1963; Eardley, 1993). Sono presenti inoltre tutte e 3 le specie attualmente riconosciute come valide, appartenenti al subgenere *Megaxylocopa* (Hurd, 1978), caso unico tra gli istituti e musei italiani.

Il numero di esemplari effettivamente presenti nel museo al tempo di A. Costa non può essere dedotto con precisione dall'analisi incrociata del materiale in cassetta, del materiale catalogato (AA.VV., 1845-1900) e da quello riportato nella

letteratura prodotta da A. Costa stesso (Tab. 3) in quanto queste tre fonti sono spesso in disaccordo su provenienza e numero di esemplari; per tali motivi solo per alcuni esemplari si è osservata corrispondenza dei tre tipi di dati (cartellino, Catalogo, pubblicazione) e solo per questi ne è stata fatta espressa citazione nel testo dei materiali (vedi sopra). A differenza del numero non elevato di esemplari, questa collezione vanta dunque una altissima varietà tassonomica e biogeografica nell'ambito degli Xylocopini, che la rende una delle più importanti collezioni mediterranee di tale tribù. Tale varietà viene ulteriormente evidenziata dai punti seguenti.

1) Nel Catalogo del museo figurano 20 esemplari non presenti nelle cassette e che quindi sono andati perduti, tra i quali vi sono almeno altre 6 specie (Tab. 2). Al n. progr. 8554 risultano 3 esemplari riportati come *Xylocopa* sp. (tab. 2) di cui però solo 1 è tuttora presente (posizione: 339 III 1): tale esemplare è stato determinato come *X. auripennis*, si trattava pertanto di 3 esemplari conspecifici (Tab. 2, 3). Con *X. (Koptortosoma) nobilis* Smith 1858, specie riportata nel Catalogo ma assente tra gli esemplari nelle cassette (Tab. 2), il numero di specie totali aumenta a 38. Sono citati inoltre 4 esemplari nel Catalogo appartenenti a 3 specie, purtroppo persi, i cui nomi non sono presenti in nessuna revisione sistematica sugli Xylocopini: *X. abdominale* Fabricius, *X. flavilabra* Latreille, *X. sticticum* Fabricius (Tab. 2). A tale riguardo è importante sottolineare che le ultime 2 specie provengono da regioni italiane, rispettivamente Piemonte e Sicilia, per cui sarebbe stato molto interessante poterli visionare in quanto A. Costa non solo era già a conoscenza del mutato status sistematico degli Xylocopini (non più genere *Apis* L. ma *Xylocopa* e *Lestis*) come si desume dal Catalogo (Vicidomini, 1997c), ma egli conosceva tutte e 3 le specie italiane di tale tribù (*X. iris*, *X. valga*, *X. violacea*) (Vicidomini, 1997b). È plausibile pertanto ritenere che queste 2 specie dovessero rappresentare nuove varietà o specie intrusive dalla Francia (esemplare piemontese) o dal nordafrica (esemplare siciliano). Dato ancor più interessante è che le 3 specie vengono citate nel Catalogo col rispettivo autore, e quindi secondo Costa stesso, già descritte prima di lui. Consultando però le opere originali di Latreille, Fabricius e Dalla Torre (1896) (citati in: Vicidomini, 1997c), non risultano Xylocopini descritti con quei nomi; se ne conclude pertanto che allo stato attuale delle conoscenze, *X. abdominale*, *X. flavilabra*, *X. sticticum* sono da considerarsi tre nuovi nomi non validi di A. Costa. Analogo discorso vale per l'esemplare disperso citato come *Xylocopa* sp. di Altamura in Puglia (Tab. 2) (nuova varietà? specie intrusiva dei Balcani?). Infine nel Catalogo viene riportato un esemplare di *Acarus* sp. (n.° progr. 22059, *acquisto del 5 febbraio 1876*) che faceva parte della collezione di preparati microscopici andata dispersa, ospitato su di un esemplare di *Xylocopa* non determinato.

2) L'esemplare di *X. dejeanii* (posizione 360 VI 6) in Catalogo viene citato come *Xylocopa* sp. con una breve nota sistematica, appena leggibile, dalla quale si evince che questo costituirebbe una varietà nuova (*praegd* var.). In base alla somiglianza delle lettere identificate ed alla provenienza del reperto è possibile

avanzare l'ipotesi che tale dicitura si riferirebbe a *X. (Koptortosoma) provida* Smith 1863, oppure la nota potrebbe addirittura indicare una nuova specie da descrivere. Sicuramente la seconda ipotesi è da considerare quale valida a causa sia della notevole diversità morfo-strutturale tra *X. provida* e *X. dejanii*, che per l'ampissimo areale (S.E. Asia, Indonesia, Filippine) su cui è distribuita *X. dejanii* con numerose varietà e sottospecie (Maa, 1938; Hurd & Moure, 1963).

3) Sono rappresentati tutti i continenti e quindi i Reami biogeografici: Australia (Sud Australia; località non specificata); Asia (Cambogia; Ceylon; Giappone; Giava; India; Isola Aru; Nuova Guinea; Persia); Europa [Italia (Abruzzo, Calabria, Campania, Piemonte, Sardegna, Sicilia, Italia meridionale); Portogallo; Russia; Spagna]; Africa (Abissinia; Egitto; Sierra Leone; Sudafrica; Africa Occidentale); America (località non specificata); America settentrionale (California; New York; Tennessee); America centrale (Messico; Cuba; località non specificata); America meridionale (Bolivia; Brasile; Colombia; Uruguay). Anche in questo caso l'estensione biogeografica viene incrementata dagli esemplari presenti in Catalogo ma assenti dalle cassette (Tab. 2) dai quali si desumono le seguenti regioni aggiuntive: Celebes, Francia, Indie (Asia meridionale), Italia (Altamura, Puglia).

Tra i 22 esemplari perduti, ben 5, appartenenti a 2 specie, sono indicati come *discaricati* o *muffiti* e quindi eliminati dal personale del museo durante la direzione F.S. Monticelli (n.° progr.: 8463; 8550; 8563; Tab. 2); tra questi di particolare interesse risulta *X. aestuans* (8463) la quale viene riportata come proveniente dalla Francia (Tab. 2); ciò è in completo contrasto con il suo areale noto, infatti questa è una specie distribuita nella regione Afro-Asiatica ed assente in Europa (Hurd & Moure, 1963). Per quanto concerne 2 altri esemplari indicati come *discaricati* (n.° progr. 41633; Tab. 2) uno è stato egualmente trovato nelle cassette (posizione: 339 II 1). Quindi gli esemplari effettivamente mancanti dalle cassette sono 16 (= 22-5-1). Nel Catalogo figura inoltre, un esemplare citato come *Xylocopa*, *Giava*, *acquisto Frohstorfer*, n.° inv. 38865, *discaricato*, il quale è però stato ritrovato in pos. 360 V 7, e sicuramente è un Apoideo non appartenente alla tribù Xylocopini (n.° progr. 42676). Confrontando il materiale presente nelle cassette con quello riportato nel Catalogo risulta evidente che alcune cassette sono andate disperse (e.g.: n.° 333; vedi Tab. 2) e che numerosi esemplari, se non tutti, sono stati spostati non solo dalla loro posizione originaria all'interno della cassetta ma anche tra le cassette, non consentendo il recupero di importanti informazioni come la successione degli esemplari nel rigo. Confrontando il materiale citato in Guiglia (1953) con quello effettivamente presente nella cassetta si evince che una femmina catturata a Colloredo è andata perduta (Tab. 3).

Numerosi esemplari sono stati immessi in collezione dopo il periodo in cui A. Costa è stato direttore (1861-1898: Maio et al., 1995), come testimoniano gli esemplari delle collezioni "Vincenzo Ragazzi", "Mario Salfi", "Ferdinando Solari" e quelli dalla spedizione sul Massiccio del Pollino (Guiglia, 1953) oltre agli esemplari presenti nelle cassette e mai catalogati, per i quali non vi sono

indicazioni riguardanti i dati di immissione nelle collezioni sui cartellini. Il ritrovamento di materiale appartenente alle collezioni Ragazzi e Solari, rende ancor più importante il Museo Zoologico di Napoli in quanto dalla letteratura è noto che queste sono conservate solo presso il Museo universitario di Modena ed i Musei civici di Genova, Milano e Verona (Poggi & Conci, 1996); è plausibile ritenere che numerosi altri esemplari, anche di altri ordini, di queste 2 collezioni storiche siano conservati nel Museo Zoologico di Napoli. Va sottolineato infine che l'autore nomenclaturale di molte specie è cambiato, per cui, in numerosi casi, quelli riportati da A. Costa in Catalogo devono essere considerati non validi.

La seconda collezione entomologica di Xylocopini dell'Università Federico II, conservata presso l'Istituto di Entomologia Agraria "F. Silvestri" di Portici, contiene 248 esemplari provenienti da 14 nazioni: Brasile, Congo Belga, Etiopia, Eritrea, Grecia, Guinea Francese, Italia (Basilicata, Calabria, Campania, Emilia Romagna, Lombardia, Marche, Molise, Puglia, Sardegna, Sicilia, Toscana, Veneto), Libia, Repubblica Dominicana, Senegal, Somalia, Spagna, Africa occidentale, oltre ad una nazione non specificata del sud-est asiatico. Viene qui di seguito fornito il quadro sinottico delle specie presenti in tale collezione (Vicidomini & Priore, 1999): *P. (P.) olivieri*, *X. (C.) iris*, *X. (K.) aestuans*, *X. (Koptortosoma) albiceps* Fabricius, 1804, *X. (K.) caffra*, *X. (Koptortosoma) citrina* Friese, 1909, *X. (Koptortosoma) imitator* Smith, 1854, *X. (Koptortosoma) modesta* Smith, 1854, *X. (K.) olivacea*, *X. (Koptortosoma) senior* Vachal, 1899, *X. (Koptortosoma) sp.*, *X. (Mesotrichia) combusta* Smith, 1854, *X. (Mesotrichia) flavorufa* (DeGeer, 1778), *X. (Mesotrichia) mixta* Radoszkowski, 1881, *X. (Neoxylocopa) grisescens* Lepeletier, 1841, *X. (Neoxylocopa) mordax* Lepeletier, 1841, *X. (Platynopoda) magnifica* (Cockerell, 1929), *X. (Xenoxycopa) chiyakensis* (Cockerell, 1908), *X. (X.) inconstans*, *X. (X.) valga*, *X. (X.) violacea*, *X. (Xylomelissa) sp.* Sono presenti quindi 14 specie in più rispetto al Museo di Napoli, di cui 2 determinate solo per il subgenere; 2 subgeneri aggiuntivi sono emersi (*Mesotrichia* Westwood, 1838; *Xylomelissa* Hurd & Moure, 1963). *Xenoxycopa* è presente complessivamente con ambedue le specie che lo costituiscono (Eardley, 1987) mentre *Platynopoda* da 3 specie su 4 note (Maa, 1938; Hurd & Moure, 1963). La varietà tassonomica quindi aumenta a 51 specie (52 inclusa *X. nobilis* presente solo in Catalogo) e 19 subgeneri di *Xylocopa* (40%) (Hurd & Moure, 1963; Eardley, 1993). Per comprendere l'importanza delle collezioni entomologiche di Xylocopini dell'Università Federico II bisogna considerare che a fronte dei 332 esemplari (348 inclusi i 16 effettivamente presenti solo nel Catalogo del Museo di Napoli) il numero di subgeneri è superato, in Italia, solo dal museo "Doria" di Genova (~ 34 subgeneri) il quale però dispone di decine di collezioni entomologiche di grandi e grandissime dimensioni e di oltre 3200 esemplari di Xylocopini, dieci volte circa quello delle collezioni universitarie napoletane (Vicidomini, in preparazione).

A fronte di tale importanza però tutti gli esemplari delle collezioni entomologiche del Museo Zoologico di Napoli versano in condizioni drammatiche

sia per il notevole grado di ammassamento che per i numerosi ed estesi danni osservati in non pochi esemplari e provocati dagli antreni (vedi anche Tab. 1), fenomeno ampiamente riscontrato anche per tutti i gruppi tassonomici presenti nelle collezioni (~ 30.000 esemplari: Poggi & Conci, 1996).

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Tabella 1
 Scheda biometrica essenziale degli
 Xylocopini del Museo Zoologico di Napoli.

POSIZIONE	LT	LAA	LAP	LM	LGC	NOTE
64 I 1	-	-	-	-	-	S
64 I 2	291	272	173	174	92	
64 I 3	168	134	98	82	47	ZM(-); M
64 I 4	-	-	-	-	-	
64 I 5	186	142	106	92	49	
64 I 6	168	139	95	86	45	
64 I 7	223	191	105	104	64	ZP(+)
64 II 1	-	206	137	116	-	C; ZA
64 II 2	-	143	94	91	-	C; ZA; P
64 II 3	271	188	125	152	70	
64 II 4	177	145	108	89	51	
64 II 5	168	134	91	87	49	M
64 II 6	175	157	106	92	48	
64 III 1	200	170	108	113	55	A
64 III 2	141	129	98	73	41	ZA(-); F(-)
64 III 3	154	145	106	81	46	
64 III 4	169	145	97	87	48	ZP(+)
64 IV 1	282	220	153	147	87	A
64 IV 2	284	245	163	142	92	F(-)
64 IV 3	249	222	147	107	84	ZP(+)
64 IV 4	248	218	151	119	81	ZP(-); F(-)
64 IV 5	216	225	137	98	78	ZP(-)
64 V 1	231	202	134	123	70	
64 V 2	249	225	153	125	84	A(-); (S.c.)
64 V 3	247	228	167	124	82	
64 V 4	228	209	125	117	69	(S.c.)
64 V 5	225	200	121	122	58	
64 VI 1	220	208	143	111	58	A(+)
64 VI 2	218	200	140	120	56	
339 I 1	288	299	163	144	80	A(-)
339 I 2	290	281	179	158	95	
339 I 3	336	259	176	172	76	
339 I 4	167	140	92	85	58	
339 II 1	289	300	168	140	83	F(-)
339 II 2	180	149	100	99	55	
339 II 3	263	238	144	133	73	A(-); F(-); T
339 II 4	287	255	176	161	94	
339 III 1	291	291	192	134	80	
339 III 2	292	271	151	150	78	
339 III 3	-	286	192	180	-	C; ZA(-)
339 IV 1	280	264	162	159	79	A(-)
339 IV 2	238	212	135	124	72	
339 V 1	255	224	148	132	60	F(-)
339 V 2	254	220	152	133	80	
339 V 3	246	195	136	125	57	
339 V 4	291	282	175	146	74	
360 I 1	315	268	175	168	97	

360 I 2	219	161	111	122	67	
360 I 3	203	153	106	118	63	
360 I 4	238	214	143	132	75	F(+)
360 I 5	267	198	134	140	74	F(-)
360 II 1	172	175	127	91	58	
360 II 2	278	242	145	141	92	A(-)
360 II 3	241	217	149	136	76	F(-)
360 II 4	167	135	102	92	52	O
360 II 5	-	187	122	130	-	C
360 III 1	219	188	122	112	65	
360 III 2	222	203	156	108	80	
360 III 3	202	189	137	103	67	A
360 III 4	187	174	118	98	58	
360 III 5	200	184	112	125	67	
360 III 6	173	156	89	96	51	F(+)
360 IV 1	163	148	96	82	52	
360 IV 2	168	155	100	88	53	
360 IV 3	143	144	97	82	46	
360 IV 4	186	169	109	87	59	
360 IV 5	204	163	112	106	66	F(+)
360 IV 6	232	210	143	126	60	ZA(-); O
360 V 1	183	149	109	93	56	
360 V 2	-	135	81	-	53	ECD
360 V 3	179	148	89	93	58	
360 V 4	155	132	85	86	55	1/2F(-)
360 V 5	123	120	82	66	55	
360 V 6	233	199	124	122	66	
360 V 7	-	-	-	-	-	
360 VI 1	214	191	132	118	70	
360 VI 2	244	200	123	124	70	F(-)
360 VI 3	210	183	120	115	56	
360 VI 4	209	200	136	111	66	
360 VI 5	191	171	103	88	59	
360 VI 6	156	147	93	63	54	
341 I 1	166	142	87	84	60	
341 I 2	166	136	95	83	60	
341 I 3	180	137	86	94	57	ZP(+)
Pollino I 1	238	215	147	120	77	
Pollino I 2	226	205	145	108	69	F(-)

Tabella 2: Esemplari assenti dalle cassette
ma citati nel Catalogo (AA.VV., 1845-1900).

(*) trattasi di tre esemplari della specie *X. auripennis* (vedi testo).

Nella colonna NOTE è stata adottata la seguente simbologia:

D discaricato (= eliminato dalla collezione);

M muffita (= esemplare ammuffito);

AS acquisto Standinger; AF acquisto Frohstorfer;

CS cambio Saussure.

VOL.; N.° PROG.; ANNO	CASS. N.°	SPECIE	N.° EXX SEX	EXX IN CASS.	LOCALITÀ	NOTE & N.° INV.
I; 8463; 1845-1872	333	<i>X. aestuans</i> Fabr.	1F	-	Francia	D, 1919
I; 8464; 1845-1872	333	<i>X. flavilabra</i> Latr.	2M	-	Piemonte	-
I; 8465; 1845-1872	333	<i>X. sticticum</i> Fabr.	1	-	Sicilia	-
I; 8466; 1845-1872	333	<i>Xylocopa</i> sp.	1	-	Brasile	in: Costa, 1862
I; 8467; 1845-1872	333	<i>X. abdominale</i> Fabr.	1	-	Natal	-
I; 8550; 1845-1872	339	<i>X. latipes</i> Fabr.	1F	-	Indie	M
I; 8552; 1845-1872	339	<i>X. latipes</i> Fabr.	2F	-	Indie	-
I; 8553; 1845-1872	339	<i>X. latipes</i> Fabr.	2M	1M	Indie	-
I; 8554; 1845-1872	339	<i>Xylocopa</i> sp. (*)	3	1F	-	-
I; 8561; 1845-1872	340	<i>X. coerulea</i> Fabr.	3	1F	Giava	-
I; 8562; 1845-1872	340	<i>X. zonata</i> Klug	2	1F	Giava	-
I; 8563; 1845-1872	340	<i>X. aestuans</i> Fabr.	1M, 1F, 1?	-	Egitto	D
I; 8567; 1845-1872	340	<i>X. tabaniformis</i> Smith	4	2M+1F	Mexico	-
II; 28508; 1881	-	<i>Xylocopa</i> sp.	1	-	Altamura	n. 15525
III; 40884; 1889-1890	-	<i>X. nobilis</i> Smith	1	-	Celebes	AS; n. 37019
III; 41633; 1890-1891	-	<i>X. latipes</i> Fabr.	2M	1M	Giava	AF; D n. 37875
III; 41745; 1890-1891	-	<i>X. virginea</i>	1F+1M	1F	Tennessee	CS

Tabella 3

Elenco Xylocopini riportati della letteratura prodotta da Achille Costra durante il periodo 1862-1866 e da Delfa Guiglia (1953).

Nella colonna PROVENIENZA viene riportata l'esatta citazione e simbologia di Costa utilizzata nei riferimenti citati.

N.	SPECIE	PROVENIENZA	N. EXX.	RIFERIMENTO
149	<i>X. barbata</i>	Brasile	1	Costa, 1862
150	<i>Xylocopa</i> ?	Brasile	4	Costa, 1862
604	<i>X. violacea</i>	Sud Italia	1	Costa, 1862
1917	<i>X. coerulea</i>	Indie	1	Costa, 1864a
1918	<i>X. collaris</i>	Indie	1	Costa, 1864a
1919	<i>Xylocopa</i> ?	Messico	1	Costa, 1864a
1920	<i>Xylocopa</i> ?	?	1	Costa, 1864a
1921	<i>X. nigrita</i>		1	Costa, 1864a
1922	<i>Xylocopa</i> ?		2	Costa, 1864a
1923	<i>X. tabaniformis</i>	Messico	3	Costa, 1864a
1924	<i>L. bombylans</i>	Australia	1	Costa, 1864a
90	<i>X. augusti</i>	Montevideo	3	Costa, 1864b
405	<i>X. aestuans</i>	Indie	2	Costa, 1866
406	<i>X. frontalis</i>	Brasile	1	Costa, 1866
407	<i>X. nigrita</i>	Cadice	2	Costa, 1866
408	<i>Xylocopa</i> ?	?	3	Costa, 1866
409	<i>Xylocopa</i> ?	?	2	Costa, 1866
410	<i>Xylocopa</i> ?	?	1	Costa, 1866
411	<i>Xylocopa</i> ?	?	2	Costa, 1866
412	<i>X. coerulea</i>	Indie Orientali	2	Costa, 1866
413	<i>X. collaris</i>	Indie	1	Costa, 1866
414	<i>Xylocopa</i> ?	Camboye	1	Costa, 1866
415	<i>Xylocopa</i> ?	Bogota	1	Costa, 1866
416	<i>L. bombylans</i>	Adelaidi	1	Costa, 1866
417	<i>Lestis</i> ?	Adelaidi	1	Costa, 1866
397	<i>X. zonata</i>	Giava	1	Costa, 1871
-	<i>X. violacea</i>	Colloreto	2	Guiglia, 1953
-	<i>X. violacea</i>	Vallone Santicelli	1	Guiglia, 1953

Uniquely passagewise connected structures

Nota di Giuseppina Anatriello *
Presentata dal socio Antonio Zitarosa
nell' adunanza del 4/12/1999

KEY WORDS: Fixed points, partial order, order topology, arcs, trees, arcwise connected spaces, dendritic spaces, orderable spaces.

Riassunto. — *In molte dimostrazioni di teoremi di punto fisso su alberi, spazi dendritici, spazi unicamente connessi per archi e su insiemi totalmente ordinati completi dotati della order topology gioca un ruolo fondamentale l' intrinseca struttura algebrica di ognuno di questi spazi.*

In questa nota introdurremo la nozione algebrica di struttura unicamente connessa per passaggi che comprende tutte quelle citate.

Vengono poi individuate alcuni tipi di spazi topologici su strutture unicamente connesse per passaggi.

Abstract. — *In many proofs of theorems of fixed point on trees, dendritic spaces, uniquely arcwise connected spaces and complete totally ordered sets provided with the order topology, the inherent algebraic structure of each of these spaces plays a relevant role.*

In this paper we will introduce the algebraic notion of uniquely passagewise connected structure which takes the all quoted algebraic structures in.

Then some topological spaces are individualized on uniquely passagewise connected structures.

1 - Introduction

In [1] (see Proposition 4.3) we proved a theorem of fixed point for a continuous function defined on a complete totally ordered set provided with the order topology. The algebraic structure of totally ordered set plays a fundamental role in the proof of the above result.

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On the other hand, in many theorems of fixed point for continuous functions on trees, dendritic spaces and uniquely arcwise connected spaces the inherent algebraic structure of these spaces has been employed (see, e.g. [4], [8], [9], [10]).

The aim of this paper is to find an algebraic structure suitable for solving the following problem:

- (P) to extend Proposition 4.3 in [1] to a class of topological spaces that takes the complete totally ordered sets provided with the order topology, the dendritic spaces and the uniquely arcwise connected spaces in.

The problem (P) will be discussed in another paper (see [2]).

The inherent partially ordered structure on the dendritic spaces and uniquely arcwise connected spaces was studied in [5], [6] by various definitions of *mod*.

Precisely in [5] (resp. [6]) a *mod* is a partially ordered set $(S, \leq)$ such that:

- α_1 . For each $x, y \in S$ there exists the infimum of $\{x, y\}$ (resp. α'_1 . There exists the minimum of S).
- α_2 . For each $x \in S$ the set $\{y \in S : y \leq x\}$ is totally ordered.
- α_3 . $(S, \leq)$ is complete (i.e. each nonempty subset of S which is bounded above has the supremum, or, equivalently, each nonempty subset of S which is bounded below has the infimum).
- α_4 . The order $\leq$ is dense.

Equivalently (see [5] and [6]), a *mod* is a pair $(S, \mathcal{P})$ such that S is a nonempty set, $\mathcal{P}$ is a nonempty set of nonempty subsets of S and the following five conditions are satisfied:

- β_1 . If $\emptyset \neq \mathcal{P}_i \subseteq \mathcal{P}$ and $\bigcap_{P \in \mathcal{P}_i} P \neq \emptyset$, then $\bigcap_{P \in \mathcal{P}_i} P \in \mathcal{P}$.
- β_2 . For each $x, y \in S$ there exists $P \in \mathcal{P}$ such that $x, y \in P$.
- β_3 . For each $P \in \mathcal{P}$ there exists only one unordered pair $\{x, y\} \subseteq S$ such that $P = \bigcap_{Q \in \mathcal{P}_{x,y}} Q$, where $\mathcal{P}_{x,y} = \{Q \in \mathcal{P} : \{x, y\} \subseteq Q\}$.

β_4 . The union Y of two meeting elements of $\mathcal{P}$ is such that, for each unordered pair $\{x, y\} \subseteq Y$, it results: $\bigcap_{Q \in \mathcal{P}_{x,y}} Q \subseteq Y$, where

$$\mathcal{P}_{x,y} = \{Q \in \mathcal{P} : \{x, y\} \subseteq Q\}.$$

β_5 . For each $P \in \mathcal{P}$, we have either $\text{card } P = 1$ or $\text{card } P > 2$.

Each totally ordered set $(T, \preceq)$ can be regarded as a mod, in a natural way, if and only if the order $\preceq$ is dense and $(T, \preceq)$ is complete (see Example 2.5).

More recently, Nikiel gave in [7] the definition of *pseudo-tree*. A pseudo-tree is a partial ordered set $(S, \leq)$ that satisfies the condition α_2 ; thus the pseudo-trees take both the totally order sets and the mods in, but this structure does not satisfy some properties useful for our aim.

Now we define the notion of *uniquely passagewise connected structure*. This structure is right for solving the problem (P). Each uniquely passagewise connected structure can be regarded as a pseudo-tree (see Corollary 4.2) and each mod as a uniquely passagewise connected structure (see Proposition 2.1).

Let X be a nonempty set, $\mathcal{P}$ be a nonempty family of nonempty subsets of X , $\mathcal{D}_X$ be the set of all unordered pairs of X , and ϕ be a function from $\mathcal{P}$ into $\mathcal{D}_X$.

We will say that the triple $(X, \mathcal{P}, \phi)$, or, more shortly, the pair (X, ϕ) , or the pair $(X, \mathcal{P})$, is a *uniquely passagewise connected structure (of support X)* if and only if the following axioms are satisfied:

p_1 . ϕ is bijective.

p_2 . $\forall P \in \mathcal{P}, \phi(P) \subseteq P$.

p_3 . Let $P_1, P_2 \in \mathcal{P}$ such that $P_1 \cap P_2 \neq \emptyset$. Then, for each $x, y \in P_1 \cup P_2$, there exists $P \in \mathcal{P}$ such that $P \subseteq P_1 \cup P_2$ and $\phi(P) = \{x, y\}$.

p_4 . Let $P_1, P_2 \in \mathcal{P}$ such that $P_1 \neq P_2$, $\text{card } \phi(P_1) = \text{card } \phi(P_2) = 2$ and $\phi(P_1) \cap \phi(P_2) = P_1 \cap P_2 \neq \emptyset$. Then $P_1 \cup P_2 \in \mathcal{P}$ and $\phi(P_1 \cup P_2) = (\phi(P_1) \cup \phi(P_2)) \setminus (\phi(P_1) \cap \phi(P_2))$.

It is easy to find examples which show that the above axioms are independent.

In section 2 some examples of uniquely passagewise connected structure are given. Each element of $\mathcal{P}$ will be called *passage*.

In sections 3-5 we introduce the notions of *passagewise connected component*, *partial order with basepoint*, *end point*, *cut point* and *branch point*. These definitions are not topological although they recall those ones that are usually given in any uniquely arcwise connected space and in connected space. Let (X, τ) be a uniquely arcwise connected space and $(X, \mathcal{P}, \phi)$ be the natural uniquely passagewise connected structure on (X, τ) (see Example 2.6), then the notion of arcwise connected component is equivalent to the notion of passagewise connected component (see Example 3.2); moreover, the topological and algebraic notions of end point, cut point and branch point are equivalent (see Example 5.1). Let (X, τ) be a dendritic space and $(X, \mathcal{P}, \phi)$ be the natural uniquely passagewise connected structure on (X, τ) (see Example 2.7), then any connected component includes a passagewise connected component (see Example 3.3); moreover, any end point of (X, ϕ) is an end point of (X, τ) , and any cut point (resp. branch point) of (X, τ) is a cut point (resp. branch point) of (X, ϕ) (see Example 5.2).

In Proposition 4.3 we obtain that, from the algebraic point of view, an arc^* is a particular kind of passage.

In sections 6 (resp. 8) the notion of *dendritic-like* (resp. *uniquely orderable-passagewise connected*) topological space is introduced. These spaces are not necessarily connected and they are a generalization, respectively, of dendritic spaces and uniquely arcwise connected spaces. Each orderable space is both dendritic-like (see Example 6.2) and uniquely orderable-passagewise connected (see Example 8.1). A topological space is dendritic if and only if it is dendritic-like and connected (see Corollary 6.1). A topological space is uniquely arcwise connected space if and only if it is uniquely orderable-passagewise connected and each of its passages is connected (see Corollary 8.3).

It is well-known that, if T is a compact connected Hausdorff space and $\text{card } T > 1$, then it is provided with at least two end points (e.g. see [11]). In section 7 we show that a compact dendritic-like topological space satisfies the same property (see Proposition 7.3).

In section 9 we give the notion of *passagewise connected component*

*We recall that an *arc* is a complete totally ordered set equipped with the order topology, provided with minimum and maximum (which are said *extremes* of the arc) and such that its order is dense. Equivalently, an arc is either a compact connected Hausdorff space provided with exactly two end points (the extremes of the arc) or a singleton. In this definition an arc need not be separable and hence need not be metrizable (e.g. see [10]). We will denote the arc of extremes a and b by $[a, b]$.

topology.

Each uniquely passagewise connected structure provided with the passagewise connected component topology is both dendritic-like and uniquely orderable-passagewise connected.

2 - Uniquely passagewise connected structures

In this section we first show that a mod can be regarded as a uniquely passagewise connected structure.

Let $(S, \mathcal{P})$ be a mod. We will denote by $\phi_{\mathcal{P}} : \mathcal{P} \rightarrow \mathcal{D}_S$ the function defined by setting for each $P \in \mathcal{P}$, $\phi_{\mathcal{P}}(P) = \{x, y\}$, where $\{x, y\}$ is the unique unordered pair of elements of S such that $P = \bigcap_{Q \in \mathcal{P}_{x,y}} Q$, where $\mathcal{P}_{x,y} = \{Q \in \mathcal{P} : \{x, y\} \subseteq Q\}$ (see Introduction, β_3). We have:

Proposition 2.1 . $(S, \mathcal{P}, \phi_{\mathcal{P}})$ is a uniquely passagewise connected structure.

Proof – Clearly, $(S, \mathcal{P})$ satisfies the axioms p_1, p_2, p_3 ; moreover, $(S, \mathcal{P})$ satisfies the axiom p_4 by the proof of Lemma 2.19 in [4]. $\square$

If $(S, \mathcal{P})$ is a mod, the uniquely passagewise connected structure $(S, \mathcal{P}, \phi_{\mathcal{P}})$ is also called *mod*.

The following example shows that a uniquely passagewise connected structure is not necessarily a mod.

Example 2.2 . Let $X = \{a, b, c, d\}$, with $\text{card } X = 4$, $\mathcal{P} = \{\{a\}, \{b\}, \{c\}, \{d\}, \{a, c\}, \{c, b\}, \{c, d\}, \{a, c, b\}, \{a, c, d\}, \{b, c, d\}\}$, $\phi : \mathcal{P} \rightarrow \mathcal{D}_X$ be the function defined by setting $\phi(\{a\}) = \{a\}$, $\phi(\{b\}) = \{b\}$, $\phi(\{c\}) = \{c\}$, $\phi(\{d\}) = \{d\}$, $\phi(\{a, c\}) = \{a, c\}$, $\phi(\{c, b\}) = \{c, b\}$, $\phi(\{c, d\}) = \{c, d\}$, $\phi(\{a, c, b\}) = \{a, b\}$, $\phi(\{a, c, d\}) = \{a, d\}$, $\phi(\{b, c, d\}) = \{b, d\}$, then $(X, \mathcal{P}, \phi)$ is a uniquely passagewise connected structure but it fails to be a mod as it does not satisfy the axiom β_5 .

Let (X, ϕ) be a uniquely passagewise connected structure. For each $x, y \in X$, the passage $\phi^{-1}(\{x, y\})$ will be called *passage of extremes x and y* and denoted by $[x, y]_{\phi}$. Moreover, we set:

$$]x, y[_{\phi} = [x, y]_{\phi} \setminus \{x, y\}, [x, y[_{\phi} =]y, x]_{\phi} = [x, y]_{\phi} \setminus \{y\}.$$

The proof of the following proposition is trivial and is omitted.

Proposition 2.3 . *Let (X, ϕ) be a uniquely passagewise connected structure and $x, y \in X$. Then $[x, y]_\phi$ is the smallest passage containing $\{x, y\}$, and $[x, x]_\phi = \{x\}$.*

We will say that a uniquely passagewise connected structure *satisfies the completeness (resp. finite completeness) property* if any nonempty intersection (resp. nonempty finite intersection) of passages is a passage.

Obviously, a uniquely passagewise connected structure is a mod if and only if it satisfies the completeness property and each of its passages has at least three elements.

Now we give other examples of uniquely passagewise connected structures.

Example 2.4 . Let $X = \mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N}_3$, with $\mathbb{N}_1, \mathbb{N}_2, \mathbb{N}_3$ mutually disjoint subsets of $\mathbb{N}^\dagger$, $\mathcal{P} = \{[n_i, +\infty[\cup [n_j, +\infty[$, with $n_i \in \mathbb{N}_i$, $n_j \in \mathbb{N}_j$, and $i, j \in \{1, 2, 3\}, i \neq j\} \cup \{[n_{i_1}, n_{i_2}]$, with $n_{i_1}, n_{i_2} \in \mathbb{N}_i$, $n_{i_1} \leq n_{i_2}$, and $i \in \{1, 2, 3\}\}$, $\phi : \mathcal{P} \rightarrow \mathcal{D}_X$ be the function defined by setting $\phi([n_i, +\infty[\cup [n_j, +\infty[) = \{n_i, n_j\}$, with $n_i \in \mathbb{N}_i$, $n_j \in \mathbb{N}_j$, and $i, j \in \{1, 2, 3\}$, with $i \neq j$, and $\phi([n_{i_1}, n_{i_2}]) = \{n_{i_1}, n_{i_2}\}$, with $n_{i_1}, n_{i_2} \in \mathbb{N}_i$, $n_{i_1} \leq n_{i_2}$, and $i \in \{1, 2, 3\}$.

Then (X, ϕ) is a passagewise connected structure not satisfying the finite completeness property. Indeed if $n_1 \in \mathbb{N}_1$, $n_2 \in \mathbb{N}_2$, $n_3 \in \mathbb{N}_3$, then $([n_1, +\infty[\cup [n_2, +\infty[) \cap ([n_1, +\infty[\cup [n_3, +\infty[) = [n_1, +\infty[\notin \mathcal{P}$.

Example 2.5 . If $(X, \leq)$ is a totally ordered set, $\mathcal{P}$ is the family of all closed intervals of X , and $\phi : \mathcal{P} \rightarrow \mathcal{D}_X$ is defined by $\phi([a, b]) = \{a, b\}$, $\forall a, b \in X$, $a \leq b$, then $(X, \mathcal{P}, \phi)$ is a uniquely passagewise connected structure and will be called the *natural uniquely passagewise connected structure on $(X, \leq)$* . (X, ϕ) satisfies the finite completeness property [†];

[†]For each $i \in \{1, 2, 3\}$ and for each $a, b \in \mathbb{N}_i$, $a \leq b$, set:

$$[a, +\infty[= \{x \in \mathbb{N}_i : x \geq a\}, [a, b] = \{x \in \mathbb{N}_i : a \leq x \leq b\}.$$

[‡]If I is finite and if $\bigcap_{i \in I} [a_i, b_i] \neq \emptyset$, then:

$$\bigcap_{i \in I} [a_i, b_i] = [\max_{i \in I} a_i, \min_{i \in I} b_i].$$

if $(X, \leq)$ is complete, then (X, ϕ) satisfies the completeness property[§]; $(X, \mathcal{P})$ is a mod if and only if the order $\leq$ is dense and $(X, \leq)$ is complete.

Example 2.6 . We recall that a topological space X is called *uniquely arcwise connected* if, for each $x, y \in X$, there exists an only arc such that x and y are extremes of the arc (e.g. see [6], p. 107).

Let (X, τ) be a uniquely arcwise connected space and $\mathcal{P}$ be the set of all arcs of X ; then the pair $(X, \mathcal{P})$ is a mod (see [7], remark 5.10, p. 46); hence, $(X, \mathcal{P}, \phi_{\mathcal{P}})$ is a mod and will be called the *natural uniquely passagewise connected structure on (X, τ)* .

Example 2.7 . We recall that a topological space connected X is called *dendritic* if, for each $x, y \in X$, with $x \neq y$, there exists $z \in X$ and two no intersecting open, namely A and B , such that $x \in A$, $y \in B$, and $X \setminus \{z\} = A \cup B$ (e.g. see [10]). Obviously a dendritic space is Hausdorff. Let (X, τ) be a dendritic space and $\tau' \subseteq \tau$ be a topology on X such that (X, τ') is an arcwise connected and dendritic (see [10], Theorem 20, p. 628); hence, by Theorem 3 in [6], (X, τ') is uniquely arcwise connected. Let $\mathcal{P}$ be the set of all arcs of (X, τ') . Then $(X, \mathcal{P}, \phi_{\mathcal{P}})$ is a mod (see Example 2.6) and will be called the *natural uniquely passagewise connected structure on (X, τ)* .

Let (X, ϕ) be a uniquely passagewise connected structure. A nonempty subset Y of X will be called *passagewise connected* if and only if, for each $a, b \in Y$, $[a, b]_{\phi} \subseteq Y$.

It is easy to prove the following result.

Proposition 2.8 . *Let (X, ϕ) be a uniquely passagewise connected structure. Then X is passagewise connected, and, for each $x, y \in X$, the sets $[x, y]_{\phi}$, $]x, y[_{\phi}$ (if it is not empty) and $]x, y[_{\phi}$ (if it is not empty) are passagewise connected. Moreover, each union of passagewise connected having nonempty intersection and each nonempty intersection of passagewise connected are passagewise connected.*

[§]If $(X, \leq)$ is complete and $\{a_i\}_{i \in I} \subseteq X$, $\{b_i\}_{i \in I} \subseteq X$ and $\bigcap_{i \in I} [a_i, b_i] \neq \emptyset$, then

$$\bigcap_{i \in I} [a_i, b_i] = [\sup_{i \in I} a_i, \inf_{i \in I} b_i].$$

Remark 2.9 . Let (X, ϕ) be a passagewise connected structure, Y be a nonempty subset of X . Set:

$$\mathcal{P}_Y = \{P \in \mathcal{P} : P \subseteq Y\},$$

then Y is passagewise connected if and only if

$$(Y, \mathcal{P}_Y, P \in \mathcal{P}_Y \rightarrow \phi(P) \in \mathcal{D}_Y)$$

is a passagewise connected structure.

3 - Passagewise connected components

Let (X, ϕ) , with $\text{card } X > 1$, be a uniquely passagewise connected structure. In this section we introduce the notion of passagewise connected component of $X \setminus \{x\}$, where $x \in X$.

Let $x \in X$ and R_x be the binary relation on $X \setminus \{x\}$ such that, for each $a, b \in X \setminus \{x\}$, it results:

$$aR_x b \iff [a, b]_\phi \subseteq X \setminus \{x\}.$$

Clearly, this relation is reflexive, symmetric, and, by axiom p_3 , transitive, so R_x is an equivalence relation. Each equivalence class induced by R_x will be called a *passagewise connected component of $X \setminus \{x\}$* and, for each $y \in X \setminus \{x\}$, we will denote the equivalence class of y by C_x^y .

Example 3.1 . Let $(X, \leq)$ be a totally ordered set and (X, ϕ) be the natural uniquely passagewise connected structure on $(X, \leq)$ (see Example 2.5). For each $x, y \in X$, with $x \neq y$, C_x^y is $\{z \in X : z < x\}$ when $y < x$, and is $\{z \in X : z > x\}$ when $y > x$.

Example 3.2 . Let X be uniquely arcwise connected space and (X, ϕ) be the natural uniquely passagewise connected structure on X (see Example 2.6). Clearly, for each $x, y \in X$, with $x \neq y$, C_x^y is the arcwise connected component of $X \setminus \{x\}$ containing y .

Example 3.3 . Let be X a dendritic space and (X, ϕ) be the natural uniquely passagewise connected structure on X (see Example 2.7). By Example 2.7, for each $x, y \in X$, with $x \neq y$, C_x^y is the arcwise connected component of $(X \setminus \{x\}, \tau')$ containing y (as by Theorem 3 in [6] in an arcwise connected dendritic space the set of all connected subsets coincides with the set of all arcwise connected subsets); so C_x^y is a connected of $(X \setminus \{x\}, \tau)$ and therefore it is included in a connected component of $X \setminus \{x\}$.

Moreover, if X is an arc $[a, b]$, with $a \neq b$, and $x, y \in X$, $x \neq y$, then $C_x^y = [a, x] \setminus \{x\}$ if $y \in [a, x[$ and $C_x^y = [b, x] \setminus \{x\}$ if $y \in [b, x[$.

The proof of the following propositions are very easy and are omitted.

Proposition 3.4 . Let $a \in X$. Then, for each $b \in X \setminus \{a\}$, $C_a^b \cup \{a\}$ is passagewise connected and if $c \in X \setminus \{a\}$, $c \neq b$, it results:

$$a \in]b, c[_\phi \iff C_a^b \neq C_a^c.$$

Moreover, if $d, f \in X$, then:

$$a \in [d, f]_\phi \iff [d, a]_\phi \cap [a, f]_\phi = \{a\}.$$

Corollary 3.5 Let $a, b \in X$, $a \neq b$. If $Y = \{x \in X : a \in]b, x[_\phi \neq \emptyset\}$, then :

$$\bigcup_{x \in Y} C_a^x = X \setminus (C_a^b \cup \{a\}).$$

4 - Partial order with basepoint

Let (X, ϕ) be a uniquely passagewise connected structure.

In this section we introduce the notion of partial order on X with basepoint p , where $p \in X$.

The binary relation $\leq_p$ on X defined by setting:

$$\forall x, y \in X, x \leq_p y \text{ if and only if } x \in [p, y]_\phi$$

will be called the *partial order on X with basepoint p* .

Clearly:

if (X, ϕ) is a mod, then, for each element p of X , the partial order on X with basepoint p is the namesake relation defined on a mod in [4], and the relation $\leq_p$ is a dense partial order (see [4]);

if $(X, \leq)$ is a totally ordered set and (X, ϕ) is the natural uniquely passagewise connected structure on $(X, \leq)$ (see Example 2.5), then, for each $p \in X$, the relation $\leq_p$ coincides with $\leq$ on C_p^y when $y > p$, and coincides with $\geq$ on C_p^y when $y < p$.

It is easy to prove the following propositions.

Proposition 4.1 . For each $p \in X$, $\leq_p$ is a partial order on X and p is the minimum of $(X, \leq_p)$. Moreover, for each $x, y \in X$, $[x, y]_\phi$ is totally ordered with respect to the relations $\leq_x$ and $\leq_y$ and $x = \min_{\leq_x} [x, y]_\phi = \max_{\leq_y} [x, y]_\phi$, and, for each $a, b \in [x, y]_\phi$, it results:

$$a \leq_x b \iff b \leq_y a.$$

Corollary 4.2 For each $p \in X$, $(X, \leq_p)$ is a pseudo-tree.

Proposition 4.3 . $Y \subseteq X$ is a passage if and only if Y is passagewise connected and there exists $p \in Y$ such that $(Y, \leq_p)$ is totally ordered and provided with the minimum and the maximum.

Now, we prove:

Proposition 4.4 . Let $x, y \in X$ and $a, b \in [x, y]_\phi$ such that $a \leq_x b$. Then the relation $\leq_x$ is just $\leq_a$ on $[a, b]_\phi$.

Proof – If either $a = x$ or $a = b$ the statement is true. Now let $x \neq a \neq b$; as $a \in [x, b]_\phi$, by Corollary 3.4, it results $[x, a]_\phi \cap [a, b]_\phi = \emptyset$. Now if $t \in [a, b]_\phi$ then $t \notin [a, x]_\phi$, and so $a \in [x, t]_\phi$ as the order relation $\leq_x$ is total on $[x, y]_\phi$. Therefore $[a, t]_\phi \subseteq [x, t]_\phi$. Hence, for each $s, t \in [a, b]_\phi$ such that $s \leq_a t$, we have $s \leq_x t$ and for each $s, t \in [a, b]_\phi$ such that $s \leq_x t$, it results $s \leq_a t$ as the order $\leq_a$ is total. $\square$

The following lemma plays a significant role in the proof of Proposition 4.6 which shows that, for each $x \in X$ the ordered sets $(X, \leq_x)$ and, for each $P \in \mathcal{P}$, $(P, \leq_x)$ are complete.

Lemma 4.5 . Let (X, ϕ) be satisfying the completeness (resp. finite completeness) property and $x \in X$. Then, each (resp. each finite) nonempty subset Y of X is provided with infimum in $(X, \leq_x)$ and, set $s = \inf_{\leq_x} Y$, it results:

$$\bigcap_{y \in Y} [x, y]_\phi = [x, s]_\phi \quad \text{and} \quad \bigcap_{y \in Y} [s, y]_\phi = \{s\}.$$

Proof – Set $\bigcap_{y \in Y} [x, y]_\phi = [u, v]_\phi$ it is easy to prove that $x \in \{u, v\}$; moreover, if $\bigcap_{y \in Y} [x, y]_\phi = [x, s]_\phi$, then $s = \inf_{\leq_x} Y$. Finally, let $v \in \bigcap_{y \in Y} [s, y]_\phi$. Then, for each $y \in Y$, it results $v \in [s, y]_\phi \subseteq [x, y]_\phi$, and hence $v \in [x, s]_\phi$. But, by Proposition 3.4, $[x, s]_\phi \cap [s, y]_\phi = \{s\}$, and so $v = s$. $\square$

Proposition 4.6 . If (X, ϕ) satisfies the completeness property and $x \in X$, then the ordered sets $(X, \leq_x)$ and, for each $P \in \mathcal{P}$, $(P, \leq_x)$ are complete.

Proof – The statement follows easily from Lemma 4.5. $\square$

5 - End points, cut points and branch points

Let (X, ϕ) be a uniquely passagewise connected structure.

An element x of X will be called an *end point* (resp. a *cut point*, a *branch point*) of (X, ϕ) or, more shortly, of X if the set Γ_x has at most one element (resp. at least two elements, at least three elements), where:

if $X \setminus \{x\} = \emptyset$ then $\Gamma_x = \emptyset$,

if $X \setminus \{x\} \neq \emptyset$ then $\Gamma_x =$ the set of all passagewise connected components of $X \setminus \{x\}$.

We recall that if (X, τ) is a uniquely arcwise connected space then an element $x \in X$ is called *end point* (resp. *cut point*, *branch point*) of (X, τ) if and only if $X \setminus \{x\}$ is arcwise connected (resp. $X \setminus \{x\}$ has at least three arcwise connected components, $X \setminus \{x\}$ has at least two arcwise connected components).

Example 5.1 If (X, τ) is a uniquely arcwise connected space and (X, ϕ) is the natural uniquely passagewise connected structure on (X, τ) , then any end point of (X, ϕ) (resp. branch point, cut point) is an endpoint (resp. a branch point, cut point) of (X, τ) and conversely (see Example 3.2).

We recall that if (X, τ) is a connected space, then an element $x \in X$ is called *end point* (resp. a *branch point*) *in the sense of connected spaces* of (X, τ) if and only if $X \setminus \{x\}$ is connected (resp. $X \setminus \{x\}$ has at least three connected components, $X \setminus \{p\}$ has at least two connected components).

Example 5.2 If (X, τ) is a dendritic space and (X, ϕ) is the natural uniquely passagewise connected structure on (X, τ) (see Example 2.7) then any end point of (X, ϕ) is an end point of (X, τ) , and any cut point (resp. branch point) of (X, τ) is a cut point (resp. branch point) of (X, ϕ) (see Example 3.3).

It is easy to prove the following:

Proposition 5.3 . *If $P \in \mathcal{P}$, then the extremes of P are end points of P and each other element of P is a cut point of P . Moreover, if $Y \subseteq X$ is passagewise connected, then each branch point of Y is a branch point of X . If $y \in Y$ is an end point of X , then y is an end point of Y .*

Each uniquely passagewise connected structure satisfies a property that recalls the definition of dendritic space. This result is proved in Proposition 5.5

The following lemma is helpful to prove Proposition 5.5.

Lemma 5.4 . *Let x be a cut point of X . Then, for each $y \in X \setminus \{x\}$, it results $X \setminus C_y^x \subseteq C_x^y$.*

Proof – On the contrary suppose that there exists a passagewise connected component of $X \setminus \{y\}$, C_y^z , such that $C_y^z \neq C_y^x$ and $C_y^z \not\subseteq C_x^y$. Then there exists a passagewise connected component of $X \setminus \{x\}$, C_x^b , with $C_x^b \neq C_x^y$, such that $C_y^z \cap C_x^b \neq \emptyset$. Now, by Proposition 3.4, the set $\{y\} \cup C_y^z \cup C_x^b$ is a passagewise connected, and $y \in \{y\} \cup C_y^z \cup C_x^b$, but $x \notin \{y\} \cup C_y^z \cup C_x^b$. So it results $\{y\} \cup C_y^z \cup C_x^b \subseteq C_x^y$, a contradiction as $C_x^b \neq C_x^y$. $\square$

Proposition 5.5 *Let x_1 and x_2 be two distinct elements of X . Then, for each $i \in \{1, 2\}$, there exist $z_i \in X$ and $X_i \subseteq X$ such that X_i is passagewise connected, $x_i \in X_i$, $X_1 \cap X_2 = \emptyset$ and $(X_1 \cup \{z_1\}) \cup (X_2 \cup \{z_2\}) = X$.*

Proof — If $]x_1, x_2[_\phi \neq \emptyset$, by Proposition 3.4, for each $z \in]x_1, x_2[_\phi$, we have $C_z^{x_1} \cap (X \setminus C_z^{x_1}) = \emptyset$ and $C_z^{x_1} \cup (X \setminus C_z^{x_1}) = X$.

If $]x_1, x_2[_\phi = \emptyset$. Now, by contradiction suppose that $x \in C_{x_1}^{x_2} \cap C_{x_2}^{x_1}$, by p_4 it results $[x, x_i] \cup [x_i, x_j] = [x, x_j] \subseteq C_{x_i}^{x_j}$, with $i, j \in \{1, 2\}$ and $i \neq j$, a contradiction. Hence $C_{x_1}^{x_2} \cap C_{x_2}^{x_1} = \emptyset$ and by Lemma 5.4, $C_{x_1}^{x_2} \cup C_{x_2}^{x_1} = X$.

□

The following two lemmas play a significant role in the proof of Proposition 5.8 which shows that each uniquely passagewise connected structure that is satisfying the finite completeness property and has no branch points can be regarded as a totally ordered set whose the set of all closed intervals is the set of all passages.

Lemma 5.6 . *Let $t \in X$; if there exist a_1, a_2, a_3 , three different elements of $X \setminus \{t\}$, such that:*

$$[t, a_i]_\phi \cap [t, a_j]_\phi = \{t\}, \text{ for each } i, j \in \{1, 2, 3\}, \text{ with } i \neq j,$$

then $C_t^{a_i} \neq C_t^{a_j}$, for each $i, j \in \{1, 2, 3\}$, $i \neq j$; therefore t is a branch point of X .

Proof — By axiom p_4 we have $[t, a_i]_\phi \cup [t, a_j]_\phi = [a_i, a_j]_\phi$, for each $i, j \in \{1, 2, 3\}$, with $i \neq j$.

By contradiction suppose that the elements a_i, a_j , with $i \neq j$, belong to the same passagewise connected component of $X \setminus \{t\}$, namely C , then:

$$t \in [a_i, a_j]_\phi \subseteq C \subseteq X \setminus \{t\},$$

a contradiction. □

Lemma 5.7 . *Let (X, ϕ) be satisfying the finite completeness property, and a, b, x be three distinct elements of X such that $a \notin]x, b[_\phi$, $b \notin]a, x[_\phi$, and $x \notin]a, b[_\phi$. Then there exists $t \in]a, b[_\phi$ such that C_t^b , C_t^x , C_t^a are mutually different.*

Proof — By Lemma 4.5 there exists a $t \in X$ such that $[x, a]_\phi \cap [x, b]_\phi = [x, t]_\phi$.

If $t = b$, then $[x, a]_\phi \cap [x, b]_\phi = [x, b]_\phi$, and hence $a \in [x, b]_\phi$, a contradiction. Therefore $t \neq b$.

A similar argument shows that $t \neq a$.

By axiom p_4 , if $t = x$ we have $[x, a]_\phi \cup [x, b]_\phi = [a, b]_\phi$, and hence $x \in]a, b[_\phi$, a contradiction. So a, b and x belong to $X \setminus \{t\}$.

Now, by Lemma 4.5, we have $[t, a]_\phi \cap [t, b]_\phi = \{t\}$ and, by Proposition 3.4, $[x, t]_\phi \cap [t, a]_\phi = \{t\}$ and $[x, t]_\phi \cap [t, b]_\phi = \{t\}$. Now from Lemma 5.6 the assertion follows. $\square$

Proposition 5.8 . *Let (X, ϕ) be satisfying the finite completeness property. If X has no branch points then there exists a total order on X such that the set of all closed intervals of X is the set of all passages.*

Proof – Let $p \in X$ and $\leq_p$ be the partial order with basepoint p .

If p is an end point, then $\leq_p$ is total order by Lemma 5.7. If p is a cut point and if C_p^1, C_p^2 are the passagewise passagewise connected of $X \setminus \{p\}$, we set for each $a, b \in X$:

$a \leq_p b \iff$ either $a, b \in C_p^1$ and $b \leq_p a$, or $a \in C_p^1$ and $b \in C_p^2$, or $a, b \in C_p^2$ and $a \leq_p b$.

Clearly, the relation $\leq_p$ is a total order and the set of all intervals of $(X, \leq_p)$ is the set of all passages. $\square$

6 - Dendritic-like topological spaces

In this section we introduced the notion of *dendritic-like* topological space.

Let (X, σ) be a topological space, we say that (X, σ) is a *dendritic-like topological space* if and only if there exists a uniquely passagewise connected structure of support X such that for each $x, y \in X$, with $x \neq y$, C_x^y is open.

Proposition 6.1 *(X, σ) is dendritic if and only if it is connected and dendritic-like.*

Proof – Let (X, σ) be dendritic and (X, ϕ) be the natural uniquely passagewise structure in (X, σ) (see Example 3.3). By Theorem 20 in [10], there exists a topology $\sigma' \subseteq \sigma$, such that (X, σ') is arcwise connected and locally connected. Thus for each $x, y \in X$, with $x \neq y$, C_x^y is an arcwise connected component of $X \setminus \{x\}$ with respect to σ' , and therefore $C_x^y \in \sigma'$, by Theorem 19 in [10] and so $C_x^y \in \sigma$. Conversely, let (X, σ) be connected and dendritic-like. For each $a, b \in X$, with $a \neq b$, it results

$]a, b[_\phi \neq \emptyset$, otherwise, if there exist a and b in X such that $]a, b[_\phi = \emptyset$, then, by the proof of Proposition 5.5, we have $X = C_a^b \cup C_b^a$ and $C_a^b \cap C_b^a = \emptyset$, a contradiction as (X, σ) is connected. Therefore even from Proposition 5.5 it follows that (X, σ) is dendritic. $\square$

Example 6.2 . Let $(X, \leq)$ be a totally ordered set, τ be the order topology on X . Clearly, (X, τ) is a dendritic-like topological space (see Example 3.1) and if the order $\leq$ is not dense then X fails to be connected (e.g. see [11], p. 197, 26G).

Proposition 6.3 . *Let (X, σ) be dendritic-like. Then each connected subset of X is passagewise connected.*

Proof — By contradiction suppose that Y is connected and not passagewise connected, then there exist $a, b \in Y$ such that $[a, b]_\phi \not\subseteq Y$. Let $y \in]a, b[_\phi \setminus Y$ and $\{C_y^{y_i}\}_{i \in I}$ be the family of all passagewise connected components of $X \setminus \{y\}$ intersecting Y , then $\text{card } \{C_y^{y_i}\}_{i \in I} \geq 2$. Set $Y_i = Y \cap C_y^{y_i}$, for each $i \in I$; then $Y = \bigcup_{i \in I} Y_i$, a contradiction as $Y \cap C_y^a \neq \emptyset \neq Y \cap C_y^b$. $\square$

Now we prove the following:

Proposition 6.4 . *Let (X, σ) be a dendritic-like topological space. Then:*

(X, σ) is Hausdorff;

for each $x, y \in X$, with $x \neq y$ the set $C_x^y \cup \{x\}$ is closed;

the relative topology induced by σ on any passage $[a, b]_\phi$ is either the order topology with respect to the partial order $\leq_a$ or finer than the order topology with respect to the partial order $\leq_a$.

Moreover, if X satisfies the finite completeness property, then, for each $a, b \in X$, any passage $[a, b]_\phi$ is closed.

Proof — Clearly, from Proposition 5.5, it follows that (X, τ) is Hausdorff. By the definition of dendritic-like topological space it follows that, for each $x, y \in X$, with $x \neq y$, the set $C_x^y \cup \{x\}$ is closed. Now let $a, b \in X$, with $a \neq b$, then, for each $x \in [a, b]_\phi$, we have $[a, b]_\phi \cap C_x^b =]x, b]_\phi$ and $[a, b]_\phi \cap C_x^a =]x, a]_\phi$. Therefore, for each $x \in [a, b]_\phi$, the sets $]x, a]_\phi$ and $]x, b]_\phi$ are open for the relative topology induced on $[a, b]_\phi$ by σ .

Since $[a, a]_\phi = \{a\}$ and $X \setminus \{a\} = \bigcup_{x \in X \setminus \{a\}} C_a^x$, then $[a, a]_\phi$ is closed, for each $a \in X$.

Let $a \neq b$, set $V_1 = \{x \in S : b \in [a, x]_\phi\}$, $V_2 = \{x \in X : a \in [b, x]_\phi\}$, $V_3 = (X \setminus (V_1 \cup V_2)) \cap]a, b[_\phi$, $V_4 = X \setminus (V_1 \cup V_2 \cup]a, b[_\phi)$.

Clearly, $X = \bigcup_{i=1}^4 V_i$ and $V_i \cap V_j = \emptyset$, with $i, j \in \{1, 2, 3, 4\}$ and $i \neq j$.

Set $\mathcal{V}_1 = \bigcup_{x \in V_1} (C_b^x \cup \{b\})$, $\mathcal{V}_2 = \bigcup_{x \in V_2} (C_b^x \cup \{b\})$, $\mathcal{V}_3 = V_3$ and $\mathcal{V}_4 = \bigcup_{x \in V_4} (C_{y_x}^x \cup \{y_x\})$, with $y_x = t \in [a, b]_\phi$ such that $[x, a]_\phi \cap [x, b]_\phi = [x, t]_\phi$ (by the proof of Lemma 5.4 there exists a unique $t \in [a, b]_\phi$ such that $[x, a]_\phi \cap [x, b]_\phi = [x, t]_\phi$), for each $x \in V_4$, then $C_{y_x}^x \neq C_{y_x}^a$ and $C_{y_x}^x \neq C_{y_x}^b$.

Clearly, we have $\mathcal{V}_1 \cap [a, b]_\phi = \{b\}$, $\mathcal{V}_2 \cap [a, b]_\phi = \{a\}$, $\mathcal{V}_3 \cap [a, b]_\phi = V_3$ and $\mathcal{V}_4 \cap [a, b]_\phi = \bigcup_{x \in V_4} \{y_x\}$.

Therefore we have $X \setminus [a, b]_\phi = (\bigcup_{i=1}^4 \mathcal{V}_i) \setminus [a, b]_\phi = \bigcup_{i=1}^4 (\mathcal{V}_i \setminus [a, b]_\phi) = (\bigcup_{x \in V_1} C_b^x) \cup (\bigcup_{x \in V_2} C_a^x) \cup (\bigcup_{x \in V_4} C_{y_x}^x)$ and so the statement is completely proved. $\square$

7 - Compact dendritic-like topological spaces

It is well-known that if (T, σ) is a compact connected Hausdorff space and $\text{card } T > 1$ then it is provided with at least two end points (e.g. see [11]). We show that a compact dendritic-like topological space satisfies the same property (see Proposition 7.3).

Let (K, σ) be a compact dendritic-like topological space.

The following lemma plays a significant role in the proof of Proposition 7.3.

Lemma 7.1 . *Let $\text{card } K > 1$, $p \in K$. Then each passagewise connected component of $K \setminus \{p\}$ is provided with at least an end point.*

To facilitate the proof of Lemma 7.1 we prove the following result.

Lemma 7.2 . *Let (X, ϕ) be a uniquely passagewise connected structure, x, p, z be three distinct cut points of X , and $z \notin C_x^p \cup \{x\}$. Set $U_x = X \setminus (C_x^p \cup \{x\})$ and $U_z = X \setminus (C_z^p \cup \{z\})$. Then $U_z \subseteq U_x$.*

Proof – On the contrary suppose that $U_z \not\subseteq U_x$. Let $a \in U_z \cap (C_x^p \cup \{x\})$, then $[a, p]_\phi \subseteq C_x^p \cup \{x\}$ and $[a, z]_\phi \subseteq U_z \cup \{z\}$. As $p \in C_z^p \cap (C_x^p \cup \{x\})$, we have $[p, z]_\phi \subseteq C_z^p \cup \{z\}$ and hence it results $[a, z]_\phi \cap [p, z]_\phi = \{z\}$. By axiom p_4 , we have $[a, p]_\phi = [a, z]_\phi \cup [p, z]_\phi$, and so $z \in C_x^p \cup \{x\}$, a contradiction. $\square$

The proof of Lemma 7.1.

Proof – Let C_p^a be a passagewise connected component of $K \setminus \{p\}$. By contradiction suppose that each element of C_p^a is a cut point of K .

Let $x \in C_p^a$ and U_x be the union of all passagewise connected components of $K \setminus \{x\}$ which are distinct from C_x^p . Then $U_x \cup \{x\} = K \setminus C_x^p$ and $U_x \neq \emptyset$, $\forall x \in C_p^a$, and moreover, by Lemma 5.4, $U_x \subseteq C_p^a$.

Set $\Omega = \{U_x \cup \{x\} : x \in C_p^a\}$ and let Ω be ordered by the inclusion relation.

Let $U = \{U_{x_i} \cup \{x_i\} : i \in I\} \subseteq \Omega$ be a totally ordered bijective family. Now each element of U is closed and therefore it is compact.

Hence $\bigcap_{i \in I} (U_{x_i} \cup \{x_i\}) \neq \emptyset$.

Let $z \in \bigcap_{i \in I} (U_{x_i} \cup \{x_i\})$. By Lemma 7.2, we have $U_z \subseteq U_{x_i}$, for each $i \in I$ such that $z \neq x_i$. It follows that $U_{x_i} \cup \{x_i\} \subseteq U_z \cup \{z\}$, for each $i \in I$. Then Ω is an inductive set and, in according to Zorn's lemma, it is provided with a minimal element, namely $U_m \cup \{m\}$.

Let $y \in U_m$, by Lemma 7.2, we have, $U_y \subseteq U_m$ and hence $U_y \cup \{y\} \subseteq U_m \subset U_m \cup \{m\}$, a contradiction. $\square$

Proposition 7.3 . *Let card $K > 1$. Then K is provided with at least two end points.*

Proof — Clearly, it follows from Lemma 7.1. $\square$

Now, we prove the following proposition which generalizes the proposition II, p. 252 of [3]:

Proposition 7.4 . *Let E be the set of end points of K . Then for each $a \in K$, it results:*

$$K = \bigcup_{e \in E} [a, e]_{\phi}.$$

Proof — Let $a \in E$. Clearly, for each $b \in K \setminus (E \cup \{a\})$, there exists $e \in E$ such that $b \in]a, e[_{\phi}$. Since $b \notin E$, by Lemma 7.1, there exists a passagewise connected component of $K \setminus \{b\}$ distinct from C_b^a , namely C_b^x , such that $C_b^x \cap E \neq \emptyset$. Let $e \in C_b^x \cap E$, by Proposition 3.4, it results $b \in]a, e[_{\phi}$. $\square$

Consequently, we have:

Proposition 7.5 . *Let card $K > 1$. Then K is a passage if and only if K is provided with exactly two end points.*

Proof — Let $a, b \in K$ be the end points of K . By Proposition 7.4 we have $K = [a, b]_{\phi}$. $\square$

8 - Uniquely orderable-passagewise connected spaces

Let (X, σ) be a topological space, $(X, \mathcal{P}, \phi)$ be a uniquely passagewise connected structure. A passage $P = [a, b]_\phi$ is said *orderable* if and only if the relative topology $\sigma|_{[a, b]_\phi}$ induced by σ on $[a, b]_\phi$ is the order topology with respect to $\leq_a$.

Let (X, σ) be a topological space. We say that (X, σ) is a *uniquely orderable-passagewise connected* if and only if there exists a uniquely passagewise connected structure $(X, \mathcal{P}, \phi)$ such that each passage is orderable.

Example 8.1 Obviously all uniquely arcwise connected spaces and all orderable spaces [¶] are uniquely orderable-passagewise connected.

Let (X, σ) be a uniquely orderable-passagewise connected. Clearly, for each passage $[a, b]_\phi \in \mathcal{P}$, $([a, b]_\phi, \sigma|_{[a, b]_\phi})$ is a dendritic-like space, and if it is an arc it is the unique arc of extremes a and b . Moreover:

Proposition 8.2 $([a, b]_\phi, \sigma|_{[a, b]_\phi})$ is connected if and only if it is an arc of extremes a and b .

Proof – By Proposition 4.6 and Proposition 4.3 to prove that if $([a, b]_\phi, \sigma|_{[a, b]_\phi})$ is connected then it is an arc of extremes a, b , it suffices to observe that an orderable space is complete and its order is dense (e.g. see [11], p. 197, 26G). Conversely, if $([a, b]_\phi, \sigma|_{[a, b]_\phi})$ is an arc then it is dendritic; therefore, by Proposition 6.1, it is dendritic-like. As an arc has exactly two end points (i.e. its extremes), then by Proposition 7.5 $[a, b]$ is the passage $[a, b]_\phi$. $\square$

Now it is easy to prove:

Corollary 8.3 . A topological space is a uniquely arcwise connected space if and only if it is a uniquely orderable-passagewise connected topological space and each of its passages is connected.

[¶]We recall that an *orderable space* is a topological space which admits a total order such that its topology is the order topology.

9 - Passagewise connected component topology

Let (X, ϕ) be a uniquely passagewise connected structure. The topology generated by the family $\{C_x^y\}_{x \in X, y \in X \setminus \{x\}}$, will be said *passagewise connected component topology* and denoted by σ_ϕ .

If $(X, \leq)$ is totally ordered and (X, ϕ) the natural uniquely passagewise connected structure on X (see Example 2.5), σ_ϕ is the order topology.

Let (D, σ) be a dendritic space. The family $\{D_x^y\}_{x \in X, y \in X \setminus \{x\}}$, where D_x^y is the connected component of $X \setminus \{x\}$ containing y , generates a topology $\sigma' \subseteq \sigma$ such that (D, σ') is dendritic, rim-finite (i.e. if each of its points has a base consisting of sets whose boundaries are finite) and locally connected (see [10], Theorem 20). If $(D, \mathcal{P}, \phi)$ is the natural uniquely passagewise connected structure on (D, σ) , then $\sigma_\phi \subseteq \sigma'$.

Let (X, ϕ) be a uniquely passagewise connected structure, and σ be a topology on X . We say that (X, σ) is *locally passagewise connected* if each of its points has a base consisting of passagewise connected sets.

Proposition 9.1 . *Let (X, ϕ) be a uniquely passagewise connected structure. Then (X, σ_ϕ) is dendritic-like and uniquely orderable-passagewise connected. Moreover, (X, σ_ϕ) is rim-finite and locally passagewise connected.*

Proof – Clearly, (X, σ_ϕ) is dendritic-like. Now, let $P \in \mathcal{P}$ and $C_x^y \in \mathcal{C}$. If either both extremes of P do not belong to C_x^y or a unique extreme of P does not belong to C_x^y and the other is x , then $C_x^y \cap P$ is the emptyset; if both extremes of P belong to C_x^y , then $C_x^y \cap P$ is P ; if a unique extreme a of P does not belong to C_x^y then $C_x^y \cap P = [a, x]_\phi$. Therefore (X, σ_ϕ) is uniquely orderable-passagewise connected.

To prove that (X, σ_ϕ) is rim-finite it suffices to observe that, for each $x, y \in X$, with $x \neq y$, the boundary of C_x^y is included in $\{x\}$, and $\bigcap_{i \in I} C_{x_i}^y$, where I is a finite set, is included in $\{x_i\}_{i \in I}$.

Finally, to prove that (X, σ_ϕ) is locally orderable-passagewise connected it suffices to observe that, by Proposition 2.8, the intersection of a family of passagewise connected sets is passagewise connected. $\square$

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PROPERTIES OF A FRACTAL INTERPOLATION SCHEME

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Abstract - It is known that a class of continuous functions $\phi : I(\subset \mathbf{R}) \rightarrow \mathbf{R}$, usually called *fractal functions*, can be defined by means of Iterated Function Systems theory. It is also known how to construct the IFS so that $\phi(x)$ interpolates given data $Y = \{(x_i, y_i)\}_{i=0}^n \subset \mathbf{R}^2$. Here, affine transformations in $\mathbf{R}^2$ of fractal interpolating functions are examined. The scheme is shown to be invariant only upon affine transformations which are lower triangular ones. As a consequence of this, invariance of the interpolating scheme under symmetry and normalization of the data is deduced. Furthermore, it is proved that the fractal interpolation scheme itself is a linear operator and it can be written in *Lagrange form*.

Riassunto - E' noto che facendo ricorso alla teoria degli Iterated Function Systems é possibile introdurre una particolare classe di funzioni continue $\phi : I(\subset \mathbf{R}) \rightarrow \mathbf{R}$, dette generalmente *funzioni frattali*, e che, in particolare, l'IFS può essere costruito in

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maniera tale che la funzione $\phi(x)$ interpoli l'insieme di punti $Y = \{(x_i, y_i)\}_{i=0}^n \subset \mathbf{R}^2$. In questa nota si studia l'effetto sull'interpolante frattale di una trasformazione affine di $\mathbf{R}^2$. Si dimostra che lo schema di interpolazione risulta invariante solo per trasformazioni affini la cui parte lineare abbia matrice triangolare inferiore. Se ne deduce l'invarianza dello schema per trasformazioni dei dati quali simmetrie e normalizzazioni. Ciò permette, infine, di provare che lo schema di interpolazione frattale é un operatore lineare e di scriverlo nella *forma di Lagrange*.

1 Introduction

Interpolation of $\mathbf{R}^2$ -data by means of smooth functions, such as polynomials or splines, is a widely studied and well settled topic. Some applications, though, require nonsmooth and irregular interpolants. Fractal functions, namely functions whose graphs are fractal sets [5], offer an adequate tool in such cases. Such functions are easily obtained as attractors of Iterated Function Systems. Their properties, though, cannot be established as easily. The aim of this paper is to establish some basic properties, such as linearity and affine invariance, the latter being especially important for graphical applications and CAGD [3].

We introduce a fractal interpolatory scheme, in the style of [1], in the following way :

Definition 1. Let $Y = \{(x_i, y_i)\}_{i=0}^n$, $n \geq 2$, $x_0 \neq x_n$, be a set of points in $\mathbf{R}^2$, and pose $\Delta x_i = x_{i+1} - x_i$. We call Y a *proper* interpolating data set if the sequence $\{\Delta x_i\}_{i=0}^{n-1}$ does not change the sign in the strong sense and has at least two nonzero elements.

With the proper data set Y and the *vertical scaling vector* $\mathbf{d} = [d_1 \dots d_n]^T$, $|d_i| < 1$, one can associate the hyperbolic IFS $\Sigma_{Y,\mathbf{d}} = \{\mathbf{R}^2; w_1, \dots, w_n\}$, introducing the affine transformations $w_i : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ given by

$$w_i(\mathbf{x}) = \begin{bmatrix} a_i & 0 \\ c_i & d_i \end{bmatrix} \mathbf{x} + \begin{bmatrix} e_i \\ f_i \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad (1)$$

with

$$\begin{aligned} a_i &= \frac{\Delta x_{i-1}}{x_n - x_0}, & c_i &= \frac{\Delta y_{i-1}}{x_n - x_0} - d_i \frac{y_n - y_0}{x_n - x_0}, \\ e_i &= x_i - a_i x_n, & f_i &= y_i - c_i x_n - d_i y_n. \end{aligned}$$

In fact, for any θ such that $0 < \theta < \min_i \{(1 - |a_i|)/(1 + |c_i|)\}$, all w_i are contractions in the norm $\|\mathbf{x}\|_\theta = |x| + \theta|y|$, therefore $\Sigma_{Y,\mathbf{d}}$ is hyperbolic.

Denoting by $\mathcal{H}(\mathbf{R}^2)$ the set of nonempty compact subsets of $\mathbf{R}^2$ and by h_θ the Hausdorff metric on $\mathcal{H}(\mathbf{R}^2)$ generated by $\|\cdot\|_\theta$, one defines the operator $W_{\Sigma, \mathbf{d}} : \mathcal{H}(\mathbf{R}^2) \mapsto \mathcal{H}(\mathbf{R}^2)$, called the *Hutchinson operator* associated to $\Sigma_{Y, \mathbf{d}}$, by posing

$$W_{\Sigma, \mathbf{d}}(B) = \cup_1^n w_i(B) \quad B \in \mathcal{H}(\mathbf{R}^2) \quad (2)$$

where $w_i(B) = \{w_i(x), x \in B\}$. If our assumptions hold, namely if Y is proper and $\max |d_i| < 1$, then $W_{\Sigma, \mathbf{d}}$ is a contraction of the complete metric space $(\mathcal{H}(\mathbf{R}^2), h_\theta)$. Furthermore, its *unique* fixed point $\Phi_{Y, \mathbf{d}} \subset \mathbf{R}^2$, called the *attractor* of $\Sigma_{Y, \mathbf{d}}$, is the graph of a continuous function $\phi_{Y, \mathbf{d}} : [x_0, x_n] \rightarrow \mathbf{R}$ interpolating the data set Y , [1],[5]. In other words we have

$$W_{\Sigma, \mathbf{d}}(\Phi_{Y, \mathbf{d}}) = \Phi_{Y, \mathbf{d}}$$

where

$$\Phi_{Y, \mathbf{d}} = \{(x, \phi_{Y, \mathbf{d}}(x)) \in \mathbf{R}^2 \mid x \in [x_0, x_n]\}$$

with

$$\phi_{Y, \mathbf{d}} \in C^0[x_0, x_n] ; \quad \phi_{Y, \mathbf{d}}(x_i) = y_i, \quad i = 1, \dots, n.$$

By the uniqueness of the attractor of a hyperbolic IFS, we can refer to $\Sigma_{Y, \mathbf{d}}$ itself as a fractal interpolatory scheme. Associated with the scheme is the operator $\Sigma_{\mathbf{x}, \mathbf{d}} : \mathbf{y} \in \mathbf{R}^{n+1} \mapsto \phi_{Y, \mathbf{d}} \in C^0$, with $[\mathbf{x}\mathbf{y}] = Y$.

2 Affine invariance of the scheme

Consider the nonsingular affine mapping $\omega : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ given by

$$\omega(\mathbf{x}) = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \mathbf{x} + \begin{bmatrix} g \\ h \end{bmatrix}, \quad p, q, r, s, g, h \in \mathbf{R}, \quad \begin{vmatrix} p & q \\ r & s \end{vmatrix} \neq 0. \quad (3)$$

Given the interpolating data set $Y = \{[x_i y_i]^T\}_{i=0}^n$, consider its image $\omega(Y) = \{\omega([x_i y_i]^T)\}_{i=0}^n$.

Definition 2. We call the mapping ω given by (3) *feasible* for a proper data set Y if $\omega(Y)$ is also a proper data set.

Notice that for any proper data set Y the set of feasible mappings is not empty. In fact a sufficient condition for ω to be feasible for Y is that $p \neq 0$ and $p \notin]\alpha, \beta[$, where $\alpha = \min\{-qm, -qM\}$ and $\beta = \max\{-qm, -qM\}$, with $m = \min\{\Delta y_i / \Delta x_i\}$ and $M = \max\{\Delta y_i / \Delta x_i\}$.

Suppose, now, that Y is proper and ω is feasible for Y . A new IFS, corresponding to the transformed data set $\omega(Y)$ and the same scaling vector $\mathbf{d}$, is $\hat{\Sigma} = \Sigma_{\omega(Y), \mathbf{d}} = \{\mathbf{R}^2; \hat{w}_1, \dots, \hat{w}_n\}$ where

$$\hat{w}_i(\mathbf{x}) = \begin{bmatrix} \hat{a}_i & 0 \\ \hat{c}_i & d_i \end{bmatrix} \mathbf{x} + \begin{bmatrix} \hat{e}_i \\ \hat{f}_i \end{bmatrix} \quad , \quad (4)$$

with

$$\begin{aligned} \hat{a}_i &= \frac{p\Delta x_{i-1} + q\Delta y_{i-1}}{p(x_n - x_0) + q(y_n - y_0)} \quad , \\ \hat{c}_i &= \frac{r\Delta x_{i-1} + s\Delta y_{i-1}}{p(x_n - x_0) + q(y_n - y_0)} - d_i \frac{r(x_n - x_0) + s(y_n - y_0)}{p(x_n - x_0) + q(y_n - y_0)} \quad , \\ \hat{e}_i &= px_i + qy_i + g - \hat{a}_i(px_n + qy_n + g) \quad , \\ \hat{f}_i &= rx_i + sy_i + h - \hat{c}_i(px_n + qy_n + h) - d_i(rx_n + sy_n + h) \quad . \end{aligned} \quad (5)$$

Obviously $\hat{\Sigma}$ is hyperbolic, since $\mathbf{d}$ is unchanged and the denominators in (5) cannot vanish because of the regularity of ω . Therefore we know that its attractor $\Phi_{\omega(Y), \mathbf{d}}$ is the graph of a continuous function $\phi_{\omega(Y), \mathbf{d}}$ interpolating the points $\{\omega([x_i y_i]^T)\}_{i=0}^n$.

The question arises about how $\Phi_{\omega(Y), \mathbf{d}}$ is related to $\omega(\Phi_{Y, \mathbf{d}})$. The existing relationship is clarified by Theorem 1, below. Notice that, from now on, being $\mathbf{d}$ fixed, we can drop it from our notation.

Theorem 1. *If $\omega(Y)$ denotes the image of the proper data set Y under a feasible transformation ω given by (3), then*

$$\omega(\Phi_Y) = \Phi_{\omega(Y)} \quad (6)$$

if and only if $q = 0$.

Proof. Consider the sequence $\{L_i\}_{i=0}^{+\infty}$ of polygonal lines such that L_0 is the polygonal interpolation of Y , $L_1 = W_{\Sigma}(L_0), \dots$, $L_k = W_{\Sigma}(L_{k-1}), \dots$, where W_{Σ} has the form (2). In the Hausdorff metric, $\Phi_Y = \lim_{k \rightarrow \infty} L_k$ by definition [5], and $\omega(\Phi_Y) = \lim_{k \rightarrow \infty} \omega(L_k)$ by the continuity of ω . On the other hand, denoting by L'_0 the polygonal interpolation of $\omega(Y)$ and posing $L'_k = W_{\hat{\Sigma}}(L'_{k-1})$, $k = 1, 2, \dots$, we have that $L'_0 = \omega(L_0)$ and that the attractor of $\hat{\Sigma}$ is $\Phi_{\omega(Y)} = \lim_{k \rightarrow \infty} L'_k$. Therefore if, for every k , the following diagram commutes,

$$\begin{array}{ccccc} L_k & \xrightarrow{\omega} & \omega(L_k) & = & L'_k \\ W_{\Sigma} \downarrow & & & & \downarrow W_{\hat{\Sigma}} \\ L_{k+1} & \xrightarrow{\omega} & \omega(L_{k+1}) & = & L'_{k+1} \end{array}$$

(namely if $\omega \circ w_i = \hat{w}_i \circ \omega, \forall i$) then from $L'_0 = \omega(L_0)$ we deduce $L'_1 = \omega(L_1)$, and at every next step $L'_k = \omega(L_k)$, so that $\Phi_{\omega(Y)} = \lim_{k \rightarrow \infty} L'_k = \lim_{k \rightarrow \infty} \omega(L_k) = \omega(\Phi_Y)$.

We prove now that the commutativity condition is, indeed, $q = 0$. Since any affine transformation is a composition of a linear transformation and a translation, we can deal with the two cases separately.

First consider the regular linear case of (3), corresponding to $g = h = 0$. In this case, by (1), we have

$$(\omega \circ w_i)(\mathbf{x}) = \omega(w_i(\mathbf{x})) = \begin{bmatrix} pa_i + qc_i & qd_i \\ ra_i + sc_i & sd_i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} e_i \\ f_i \end{bmatrix}$$

which yields

$$(\omega \circ w_i)(\mathbf{x}) = \begin{bmatrix} (pa_i + qc_i)(x - x_n) + qd_i(y - y_n) + px_i + qy_i \\ (ra_i + sc_i)(x - x_n) + sd_i(y - y_n) + rx_i + sy_i \end{bmatrix}.$$

Likewise, by (4) we have,

$$(\hat{w}_i \circ \omega)(\mathbf{x}) = \begin{bmatrix} \hat{a}_i[p(x - x_n) + q(y - y_n)] + px_i + qy_i \\ \hat{c}_i[p(x - x_n) + q(y - y_n)] + d_i[r(x - x_n) + s(y - y_n)] + rx_i + sy_i \end{bmatrix}.$$

For any $i = 1, \dots, n$, the difference $\delta_i(\mathbf{x}) = (\omega \circ w_i - \hat{w}_i \circ \omega)(\mathbf{x})$ is given by

$$\delta_i(\mathbf{x}) = \begin{bmatrix} [p(a_i - \hat{a}_i) + qc_i](x - x_n) + q(d_i - \hat{a}_i)(y - y_n) \\ [r(a_i - d_i) + sc_i - p\hat{c}_i](x - x_n) - q\hat{c}_i(y - y_n) \end{bmatrix}. \quad (7)$$

And (6) is equivalent to $\delta_i(\mathbf{x}) = [0 \ 0]^T$, identically, $\forall i$, which holds if and only if

$$p(a_i - \hat{a}_i) + qc_i = 0, \quad q(d_i - \hat{a}_i) = 0, \quad \forall i \quad (8)$$

and

$$r(a_i - d_i) + sc_i - p\hat{c}_i = 0, \quad q\hat{c}_i = 0, \quad \forall i. \quad (9)$$

Therefore, $q = 0$ implies $\delta_i(\mathbf{x}) = [0 \ 0]^T$, $\forall i$, since $\hat{a}_i|_{q=0} = a_i$ and $\hat{c}_i|_{q=0} = r(a_i - d_i)/p + sc_i/p$.

Suppose now $q \neq 0$. Then, by the second equation in (8) and (9), $\hat{a}_i = d_i$ and $\hat{c}_i = 0$, so that (8) and (9) reduce to

$$p(a_i - d_i) + qc_i = 0, \quad r(a_i - d_i) + sc_i = 0.$$

The above system has nontrivial solution if and only if

$$\begin{vmatrix} p & q \\ r & s \end{vmatrix} = 0,$$

which, by supposition, never occurs. So, it must be $q = 0$.

Consider now the translation case, $q = r = 0$, $p = s = 1$, $|g| + |h| > 0$ in (13), that is

$$\omega(\mathbf{x}) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} g \\ h \end{bmatrix}, \quad |g| + |h| > 0.$$

In this case $\hat{a}_i = a_i$ and $\hat{c}_i = c_i$, while

$$\hat{e}_i = e_i + (1 - a_i)g, \quad \hat{f}_i = f_i + (1 - d_i)h - c_i g,$$

which yields

$$\hat{w}_i(\mathbf{x}) = w_i(\mathbf{x}) + \begin{bmatrix} (1 - a_i)g \\ (1 - d_i)h - c_i g \end{bmatrix} = \begin{bmatrix} a_i(x - g) \\ c_i(x - g) + d_i(y - h) \end{bmatrix} + \begin{bmatrix} g \\ h \end{bmatrix}.$$

Therefore

$$\hat{w}_i(\mathbf{x}) = (w_i \circ \omega^{-1})(\mathbf{x}) + \begin{bmatrix} g \\ h \end{bmatrix} = (\omega \circ w_i \circ \omega^{-1})(\mathbf{x}),$$

which again leads to $\hat{w}_i \circ \omega = \omega \circ w_i$ and therefore to $\omega(\Phi_Y) = \Phi_{\omega(Y)}$. $\square$

Remark 1. We note, at this point, that if ω is nonsingular, then $q = 0$ implies $p \neq 0$, which, in turn, guarantees feasibility of ω .

Remark 2. Also we would like to mention, here, that an alternative proof of Theorem 1 can be given, based on the functional equation of Read-Bajraktarevich type

$$\phi(x) = v_i(u_i^{-1}(x), \phi(u_i^{-1}(x))), \quad x \in [x_{i-1}, x_i], \quad 1 = 1 \dots, n$$

satisfied by the fractal interpolant. We gave the proof in [4].

3 Symmetries and data normalization

Since symmetries with respect to coordinate axes and to the origin are linear transformations with diagonal coefficients matrix, a useful consequence of Theorem 1 is invariance of the scheme $\Sigma_{Y,d}$ under the basic symmetry transformations. The most general result that can be stated about symmetry invariance is given in Corollary 1.

For any $S \subset \mathbf{R}^2$, let us denote by $S^c = \{(2c - x, y) \mid (x, y) \in S\}$ its symmetric image with respect to the fixed line $\{(c, y) \mid y \in \mathbf{R}\}$. Likewise, denote by $S_c = \{(x, 2c - y) \mid (x, y) \in S\}$ the symmetric image of $S \subset \mathbf{R}^2$ with respect to the fixed line $\{(x, c) \mid x \in \mathbf{R}\}$. With this notation

Corollary 1 (*Symmetry*). If Y is a proper data set, and $c \in \mathbf{R}$,

$$\Phi_{Y^c} = (\Phi_Y)^c \quad \text{and} \quad \Phi_{Y_c} = (\Phi_Y)_c.$$

Proof. Is straightforward from Theorem 1. Letting $p = -1$, $q = 0$, $r = 0$, $s = 1$, $g = 2c$ and $h = 0$ in the first case, and $p = 1$, $q = 0$, $r = 0$, $s = -1$, $g = 0$ and $h = 2c$ in the second case. $\square$

Corollary 2 (*Normalized mesh*). The fractal interpolatory scheme $\Sigma_{Y,d}$ is invariant under normalization of the data mesh.

Proof. The transformation ω_1 given by (3) with the particular choice of the coefficients $p = 1/(x_n - x_0)$, $q = 0$, $g = -x_0/(x_n - x_0)$, $r = h = 0$, $s = 1$, takes the stripe $[x_0, x_n] \times \mathbf{R}$ into the stripe $[0, 1] \times \mathbf{R}$, accomplishing normalization of the interpolation mesh. If mesh normalization must be performed *preserving the aspect ratio* of the data set Y , this is done by an affine transformation ω_2 of the type (3), with coefficients $p = s = 1/(x_n - x_0)$, $q = 0$, $g = -x_0/(x_n - x_0)$, $r = 0$, $h = -y_0/(x_n - x_0)$. Both ω_1 and ω_2 satisfy the condition of Theorem 1, therefore they preserve the interpolation scheme. $\square$

Finally, it is often convenient to restrict to data sets that fit into the unit square $\mathbf{U} = [0, 1] \times [0, 1]$. Now, any proper data set Y can be normalized into a data set $\omega(Y) \subset \mathbf{U}$ by means of the affine transformation ω having coefficients $p = 1/(x_n - x_0)$, $q = 0$, $g = -x_0/(x_n - x_0)$, $r = 0$, and $s = 1/(y_M - y_m)$, $h = -y_0/(y_M - y_m)$, where $y_M = \max\{y_i\}_{i=0}^n$ and $y_m = \min\{y_i\}_{i=0}^n$. Such a transformation ω obviously satisfies the condition of Theorem 1, so we can state that

Corollary 3 (*Normalized data set*). Normalization of the data set Y does not affect the fractal interpolatory scheme.

4 Linearity property

In $\mathcal{H}(\mathbf{R}^2)$, let us define a particular type of scalar multiplication by

$$\lambda S = \{(x, \lambda y) \mid (x, y) \in S\}, \quad S \in \mathcal{H}(\mathbf{R}^2), \quad \lambda \in \mathbf{R}, \quad (10)$$

and a particular type of addition by

$$S_1 + S_2 = \{(x, y_1 + y_2) \mid (x, y_1) \in S_1, (x, y_2) \in S_2\}, \quad S_1, S_2 \in \mathcal{H}(\mathbf{R}^2). \quad (11)$$

Note that scalar multiplication by $s \in \mathbf{R}$ in $\mathcal{H}(\mathbf{R}^2)$ corresponds to pointwise affine transformation in $\mathbf{R}^2$ by means of (3) with the coefficients $p = 1$, $q = r = g = h = 0$ and $s \neq 0$. Therefore, again by Theorem 1,

Corollary 4 (*Homogeneity*). The interpolating scheme $\Sigma_{Y,\mathbf{d}}$ is homogenous in the sense that for any proper data set Y and for any $\lambda \in \mathbf{R}$

$$\Phi_{\lambda Y} = \lambda \Phi_Y,$$

where the scalar multiplication is understood as defined by (10).

We prove, now, that the scheme also has additivity property. Remember that indication of the vertical scaling vector $\mathbf{d}$ is omitted since this is the same for all the involved IFS's.

Theorem 2 (*Additivity*). Let Y_1 and Y_2 be two proper data sets defined on the same mesh $\{x_i\}_{i=0}^n$. Consider the corresponding IFS's, Σ_{Y_1} and Σ_{Y_2} with the same vertical scaling vector. Then,

$$\Phi_{Y_1+Y_2} = \Phi_{Y_1} + \Phi_{Y_2},$$

where addition is taken according to (11).

Proof. Let $\Sigma_{Y_j} = \{\mathbf{R}^2; w_1^{(j)}, \dots, w_n^{(j)}\}$ be the IFS associated with the data set $Y_j = \{(x_i, y_i^{(j)})\}_{i=0}^n$, for $j = 1, 2$. Denote the x and y components of $w_i^{(j)}(\mathbf{x})$ by $u_i^{(j)}(x)$ and $v_i^{(j)}(x, y)$, respectively. Consider the data set $Y = Y_1 + Y_2 = \{(x_i, (y_i^{(1)} + y_i^{(2)}))\}_{i=0}^n$, and let $\Sigma_Y = \{\mathbf{R}^2; w_1, \dots, w_n\}$ be the associated IFS, containing affine mappings w_i having decomposition (u_i, v_i) . It is easy to see, by direct computation of the coefficients, that

$$u_i(x) = u_i^{(1)}(x) = u_i^{(2)}(x), \quad (12)$$

and

$$v_i(x, y' + y'') = v_i^{(1)}(x, y') + v_i^{(2)}(x, y''). \quad (13)$$

Similarly as in the proof of Theorem 1, with obvious notation, we can introduce the sequences of piecewise linear interpolants $\{L_k^{(1)}\}_{k=0}^\infty$, $\{L_k^{(2)}\}_{k=0}^\infty$, and $\{L_k\}_{k=0}^\infty$. Using (12) and (13) we see that $L_k = L_k^{(1)} + L_k^{(2)}$ holds, in the sense of (11), for any k . From this we deduce that, in $(\mathcal{H}(\mathbf{R}^2), h_\theta)$,

$$\Phi_{Y_1} + \Phi_{Y_2} = \lim_{k \rightarrow \infty} L_k^{(1)} + \lim_{k \rightarrow \infty} L_k^{(2)} = \lim_{k \rightarrow \infty} (L_k^{(1)} + L_k^{(2)}) = \lim_{k \rightarrow \infty} L_k = \Phi_Y$$

which completes our proof. $\square$

The linearity of the operator $\Sigma_{\mathbf{x},\mathbf{d}} : \mathbf{R}^{n+1} \rightarrow C^0$ is a straightforward consequence of Corollary 4 and Theorem 2. In fact

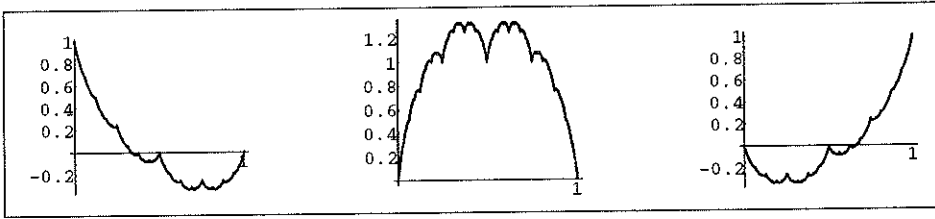


Figure 1: Fractal basis functions over three points, scaling factors = 0.5

Corollary 5 (Linearity). For any pair of proper data sets Y_1 and Y_2 and any real constants λ and μ

$$\Phi_{\lambda Y_1 + \mu Y_2} = \lambda \Phi_{Y_1} + \mu \Phi_{Y_2}.$$

Furthermore, because of the linearity property, the interpolation scheme Σ_Y can be put in *Lagrange form*. Let $Y = \{(x_i, y_i)\}_{i=0}^n$ be a proper set of interpolating data. For $j = 0, \dots, n$, let Φ_j denote the attractor of the IFS Σ_{Y_j} , associated with the data set $Y_j = \{(x_i, \delta_{ij})\}_{i=0}^n$, where δ_{ij} is Kronecker's delta. This is the graph of a continuous function $\varphi_j(x)$ such that $\varphi_j(x_i) = \delta_{ij}$. By Corollary 5

$$\Phi_Y = \sum_{j=0}^n y_j \Phi_j,$$

And, of course, the same decomposition that applies to the graphs also applies to the functions, therefore

Corollary 6 (Lagrange form). The fractal function $\phi(x)$ interpolating the data set $Y = \{(x_i, y_i)\}_{i=0}^n$ admits the decomposition

$$\phi(x) = \sum_{j=0}^n y_j \varphi_j(x), \quad x \in [x_0, x_n],$$

where $\{\varphi_j(x)\}_{j=0}^n$ are the fractal functions interpolating the *canonical data sets* $Y_j = \{(x_i, \delta_{ij})\}_{i=0}^n$ on the given mesh.

In this sense, for a fixed mesh $\{x_i\}_{i=0}^n$ and a fixed vertical scaling vector $\mathbf{d}$, the set of functions $\{\varphi_j\}_{j=0}^n$ is a basis in the space of fractal functions interpolating on that mesh with the prescribed scaling vector. Figures 1 and 2 illustrate different fractal bases, corresponding to a normalized and regular mesh, with different number of nodes and different scaling vectors.

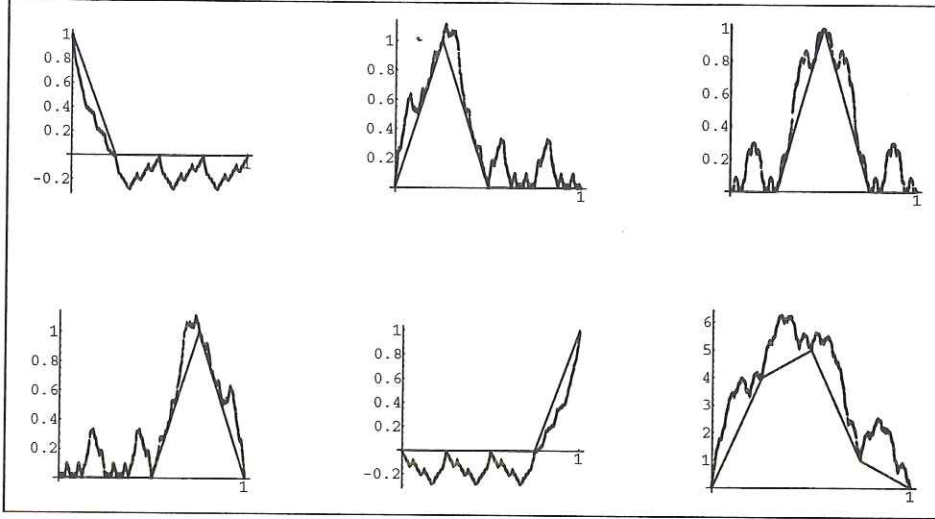


Figure 2: Fractal basis functions over five points, scaling factors = 0.2 , and the blended fractal function interpolating data from Ex. 1

5 Examples

The following examples illustrate the theory.

Example 1. Consider the data set $Y = \{(0, 0), (3, 4), (5, 5), (7, 1), (10, 0)\}$ and the vertical scaling vector $\mathbf{d} = [0.3 \ -0.2 \ 0.3 \ 0.2]^T$. The graph of the corresponding fractal interpolating function is shown in Figure 3(a), marked by F_Y . Also consider a linear transformation ω defined by

$$\omega(\mathbf{x}) = \begin{bmatrix} 0.9192 & 0.2257 \\ 0.7713 & 1.2803 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The graphs $\omega(F_Y)$ and $F_{\omega(Y)}$ are also shown. Notice that they are translated upwards a little bit, for the sake of clarity. Actually ω is a linear contraction, so $(0, 0)$, which is also the first interpolation node, is its fixed point. It is evident from the picture that $\omega(F_Y)$ and $F_{\omega(Y)}$ differ from each other, which is to be expected because ω does not fulfil the conditions of Theorem 1.

Example 2. For the data $Y = \{(0, 0), (3, 4), (7, 1), (10, 0)\}$ and $\mathbf{d} = [0.5 \ -0.28 \ 0.3]^T$ and for the same transformation ω as in Example 1, the graphs of $\omega(F_Y)$ and $F_{\omega(Y)}$ are displayed together in Figure 3(b). The difference between these is also evident.

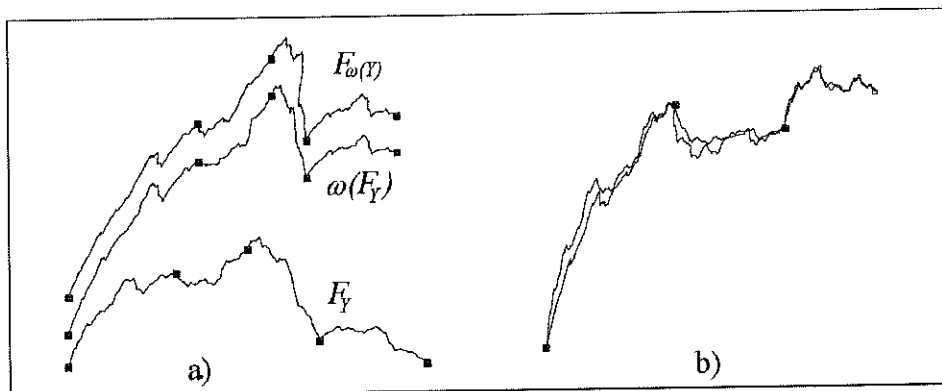


Figure 3: Illustration of Example 1 (a) and Example 2 (b)

Example 3. For the same Y and $\mathbf{d}$ as in Example 1, but for

$$\omega(\mathbf{x}) = \begin{bmatrix} 1 & -0.2257 \\ 0 & 1.2803 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

the graphs of $\omega(F_Y)$ and $F_{\omega(Y)}$ again differ, as shown in Figure 4(a).

Example 4 (Invariant case). With the same interpolating data and vertical scaling vector as in Example 1, consider linear mapping

$$\omega(\mathbf{x}) = \begin{bmatrix} 0.866 & 0 \\ 0.5 & 1.2803 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

which satisfies conditions of Theorem 1. As expected, ω maps F_Y to the graph of the fractal function interpolating the transformed data set $\omega(Y)$. This is shown in Figure 4(b). Unfortunately, it is difficult to make it evident that the two graphs coincide, by a black and white picture on paper. The software documented in [2] allows for interactive sessions and for the use of colour, so that, for example, to verify the affine invariance property the user can observe the black graph $\omega(F_Y)$ being gradually covered in red as $F_{\omega(Y)}$ is slowly drawn on the screen.

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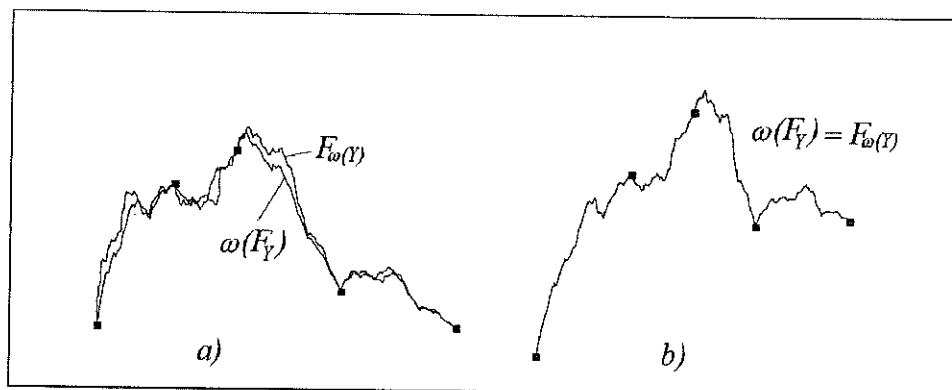


Figure 4: Illustration of Example 3 (a) and Example 4 -invariant case- (b)

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APPENDICE

NORME PER LA STESURA DI UN DATTILOSCRITTO

Prima pagina.

Si lasciano vuoti quattro righi dall'alto.

Titolo: **Grassetto, Times New Roman (TNR) 13 con testo centrato.**

**La Secca delle Formiche di Vivara
(canale d'Ischia, Campania)
Osservazioni Geomorfologico-strutturali e Faunistiche**

A distanza di 1 rigo digitare gli autori della nota con **caratteri TNR 11:**

Nota di Vincenzo Rosso¹, Nicola Verde² e Pasquale Bianco^{1,*}

A distanza di 0.5 rigi digitare il nome del presentatore e la data dell'adunanza con **caratteri TNR 9:**

Presentata dal socio Michele Russo
(Adunanza del 6 febbraio, 1999)

A distanza di 1.5 rigi digitare in lingua inglese le Key words con **caratteri TNR 9** come indicato:

Key words: stratigraphy, island, DNA, proteinase, etc.

A distanza di 0.5 rigi digitare, come indicato, due riassunti con **caratteri TNR 9** di cui uno in italiano e l'altro in inglese:

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Hudson, T.H. and Grillo, F.G. (1993) *Reactivation of an inactive human X chromosome*. Science, **230**, 157-163.

Abrahams, R. and Marsden, J. (1978) *Foundation of mechanics*. Benjamin, New York, NY

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¹ Dipartimento 1, indirizzo 1; Dipartimento 2, indirizzo 2 e così per i vari numeri che nel titolo distinguono gli autori.

* Author to whom correspondence should be addressed (Autore a cui indirizzare la corrispondenza)

Esempio di prima pagina

**Analisi sismologiche fatte
nella terra d'Otranto nel decennio 1970-1980**
Nota di Vincenzo Rosso¹, Nicola Verde² e Pasquale Bianco^{1*}

Presentata dal socio Michele Russo
(Adunanza del 6 febbraio, 1999)

Key words: : stratigraphy, Apulia, protein, hemoglobin etc.

Abstract - Abstract written in English language, in Times New Roman 9, justified; abstract written in English language, in Times New Roman 9, justified; abstract written in English language, in Times New Roman 9, justified; abstract written in English language, in Times New Roman 9, justified;

Riassunto - Scritto giustificato in Times New Roman 9, in italiano; scritto giustificato in Times New Roman 9, in italiano; scritto giustificato in Times New Roman 9, in italiano; scritto giustificato in Times New Roman 9, in italiano; scritto giustificato in Times New Roman 9, in italiano; scritto giustificato in Times New Roman 9, in italiano.

1 - INTRODUCTION

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